View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by CERN Document Server

CERN-TH/97-306
UCLA/97/TEP/24
ROM2F-97/37
hep-th/9711059

Tensor and Vector Multiplets in Six-Dimensional Supergravity

Sergio Ferrara[†,‡], Fabio Riccioni[*,1] and Augusto Sagnotti[*]

[†] *Theory Division, CERN*
CH-1211 Geneva 23, SWITZERLAND

[‡] Department of Physics and Astronomy,
University of California, Los Angeles, CA 90024 U.S.A.

[*] *Dipartimento di Fisica, Università di Roma "Tor Vergata"*
I.N.F.N. - Sezione di Roma "Tor Vergata",
Via della Ricerca Scientifica , 1 00133 Roma ITALY

Abstract

We construct the complete coupling of $(1, 0)$ supergravity in six dimensions to n tensor multiplets, extending previous results to all orders in the fermi fields. We then add couplings to vector multiplets, as dictated by the generalized Green-Schwarz mechanism. The resulting theory embodies factorized gauge and supersymmetry anomalies, to be disposed of by fermion loops, and is determined by corresponding Wess-Zumino consistency conditions, aside from a quartic coupling for the gaugini. The supersymmetry algebra contains a corresponding extension that plays a crucial role for the consistency of the construction. We leave aside gravitational and mixed anomalies, that would only contribute to higher-derivative couplings.

CERN-TH/97-306
November 1997

[1] I.N.F.N. Fellow

1 Introduction

One of the most striking features of perturbative superstring theory in ten dimensions is the absence of anomalies. In the type-IIB theory this is realized by miraculous cancellations between various contributions [1], while in the type-I and heterotic theories the Green-Schwarz mechanism [2] generates anomalous couplings that exactly cancel the contributions of fermion loops, once one restricts the gauge group to be $SO(32)$ for the type I theory and $SO(32)$ or $E_8 \times E_8$ for the heterotic theory. All these $N = 1$ theories are very interesting, since they can be naturally compactified to rich spectra of $N = 1$ theories in four dimensions. In this context, an interesting intermediate step is the study of $(1, 0)$ vacua in six dimensions, since in these compactifications the absence of anomalies is a strong restriction on the low-energy physics.

The massless representations of $(1, 0)$ supersymmetry in six dimensions, labeled by their $SU(2) \times SU(2)$ representations, are the gravity multiplet $((1, 1) + 2(1, \frac{1}{2}) + (1, 0))$, the tensor multiplet $((0, 1)+2(0, \frac{1}{2})+(0, 0))$, the vector multiplet $((\frac{1}{2}, \frac{1}{2})+2(\frac{1}{2}, 0))$, and the hypermultiplet $(2(0, \frac{1}{2}) + 4(0, 0))$. [3] considered pure $(1, 0)$ supergravity, and in [4] $(1, 0)$ supergravity was coupled to an arbitrary number of tensor multiplets to lowest order in the fermi fields, while [5] considered the case of a single tensor multiplet and an arbitrary number of hypermultiplets. It was then found [6] that the model in [4] can be coupled to Yang-Mills multiplets in a way determined by the residual gauge and gravitational anomalies. The relation to the supersymmetry anomaly was elucidated in [7], to lowest order in the fermi fields, and the resulting coupling to hypermultiplets was then partly constructed in [8].

Letting n_T, n_V and n_H denote the numbers of tensor, vector and hypermultiplets, the condition that the term trR^4 be absent in the anomaly polynomial [9],

$$n_H - n_V + 29n_T = 273 \quad , \tag{1.1}$$

allows a large number of possible vacua. Perturbative heterotic vacua in six dimensions can be obtained by orbifold compactifications or by compactifications on smooth $K3$ manifolds with instanton backgrounds. Anomaly cancellation requires that the total instanton number be 24, and these vacua include a single tensor multiplet, as one can easily see reducing the ten dimensional low-energy theory. The situation is quite different in perturbative six-dimensional type I vacua since, as suggested in [10], these models are determined by a parameter space orbifold (orientifold) construction, and this naturally allows several tensor multiplets [11]. The residual anomaly polynomial in general does not factorize, and several antisymmetric tensors contribute to the cancellation in a generalized Green-Schwarz mechanism [6]. Moreover, the low-energy supergravity exhibits singularities in the moduli space of tensor multiplets, corresponding to infinite gauge coupling constants [6]. These singularities have attracted some interest, since they reflect the presence in the vacuum of string excitations with vanishing tension [12], and signal a new kind of phase transition [13]. The conjectured type I - heterotic duality [14] relates these peculiar perturbative type-I vacua to corresponding non-perturbative heterotic vacua.

In this paper we construct the complete $(1,0)$ supergravity coupled to tensor and vector multiplets. This theory contains (reducible) gauge and supersymmetry anomalies induced by tensor couplings, that here are completely determined solving Wess-Zumino consistency conditions. In Section 2 we construct minimal supergravity coupled to n tensor multiplets, thus completing [3, 4] to all orders in the fermi fields. In Section 3 we include all additional couplings to vector multiplets, thus completing [6, 7]. Some of the higher-order fermion couplings were previously introduced in [8], where couplings to hypermatter were also considered. Section 4 is devoted to a discussion of our results, while the Appendix collects some details on our notation and a number of identities used in our derivations. While this work was being typed, a lagrangian superspace formulation of the theory with tensor multiplets only was presented in [15].

2 Minimal Supergravity in Six Dimensions Coupled to n Tensor Multiplets

In this Section we describe minimal $(1,0)$ six-dimensional supergravity coupled to n tensor multiplets. Simple supersymmetry in six dimensions is generated by an $Sp(2)$ doublet of chiral spinorial charges Q^a $(a = 1, 2)$, obeying the symplectic Majorana condition

$$Q^a = \epsilon^{ab} C \bar{Q}_b^T \quad , \tag{2.1}$$

where ϵ^{ab} is the $Sp(2)$ antisymmetric invariant tensor. Since all fermi fields appear as $Sp(2)$ doublets, from now on we will mostly use Ψ to denote a doublet Ψ^a. Further details on this notation may be found in the Appendix.

Let us begin by reviewing the work of Romans [4]. The theory includes the vielbein ${e_\mu}^a$, a left-handed gravitino Ψ_μ, $(n+1)$ antisymmetric tensors $B^r_{\mu\nu}$ $(r = 0, ..., n)$ obeying (anti)self-duality conditions, n right-handed "tensorini" χ^m $(m = 1, ..., n)$, and n scalars. The scalars parameterize the coset space $SO(1,n)/SO(n)$, and are thus associated to the $SO(1,n)$ matrix $(r = 0, ...n)$

$$V = \begin{pmatrix} v_r \\ x_r^m \end{pmatrix} \quad , \tag{2.2}$$

whose matrix elements satisfy the constraints

$$\begin{aligned} & v^r v_r = 1 \quad , \\ & v_r v_s - x_r^m x_s^m = \eta_{rs} \quad , \\ & v^r x_r^m = 0 \quad . \end{aligned} \tag{2.3}$$

Defining

$$G_{rs} = v_r v_s + x_r^m x_s^m \quad , \tag{2.4}$$

the tensor (anti)self-duality conditions can be succinctly written

$$G_{rs} H^{s\mu\nu\rho} = \frac{1}{6e} \epsilon^{\mu\nu\rho\alpha\beta\gamma} H_{r\alpha\beta\gamma} \quad , \tag{2.5}$$

where $H^r_{\mu\nu\rho} = 3\partial_{[\mu} B^r_{\nu\rho]}$. These relations only hold to lowest order in the fermi fields, and imply that $v_r H^r_{\mu\nu\rho}$ is self dual, while the n tensors $x_r^m H^r_{\mu\nu\rho}$ are antiself dual, as one can see using eqs. (2.3). The divergence of eq. (2.5) yields the second-order tensor equation

$$D_\mu (G_{rs} H^{s\mu\nu\rho}) = 0 \tag{2.6}$$

while, to lowest order, the fermionic equations are

$$\gamma^{\mu\nu\rho} D_\nu \Psi_\rho + v_r H^{r\mu\nu\rho} \gamma_\nu \Psi_\rho - \frac{i}{2} x_r^m H^{r\mu\nu\rho} \gamma_{\nu\rho} \chi^m + \frac{i}{2} x_r^m \partial_\nu v^r \gamma^\nu \gamma^\mu \chi^m = 0 \tag{2.7}$$

and

$$\gamma^\mu D_\mu \chi^m - \frac{1}{12} v_r H^{r\mu\nu\rho}\gamma_{\mu\nu\rho}\chi^m - \frac{i}{2} x_r^m H^{r\mu\nu\rho}\gamma_{\mu\nu}\Psi_\rho - \frac{i}{2} x_r^m \partial_\nu v^r \gamma^\mu \gamma^\nu \Psi_\mu = 0 \quad . \tag{2.8}$$

Varying the fermi fields in them with the supersymmetry transformations

$$\begin{aligned}
&\delta e_\mu{}^a = -i(\bar{\epsilon}\gamma^a \Psi_\mu) \quad , \\
&\delta B^r_{\mu\nu} = i v^r (\bar{\Psi}_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2} x^{mr}(\bar{\chi}^m \gamma_{\mu\nu}\epsilon) \quad , \\
&\delta v_r = x_r^m (\bar{\epsilon}\chi^m) \\
&\delta \Psi_\mu = D_\mu \epsilon + \frac{1}{4} v_r H^r_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon \quad , \\
&\delta \chi^m = \frac{i}{2} x_r^m \partial_\mu v^r \gamma^\mu \epsilon + \frac{i}{12} x_r^m H^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon \quad ,
\end{aligned} \tag{2.9}$$

generates the bosonic equations, using also eqs. (2.5) and (2.6). Thus, the scalar field equation is

$$x_r^m D_\mu(\partial^\mu v^r) + \frac{2}{3} x_r^m v_s H^r_{\alpha\beta\gamma} H^{s\alpha\beta\gamma} = 0 \quad , \tag{2.10}$$

while the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \partial_\mu v^r \partial_\nu v_r - \frac{1}{2} g_{\mu\nu}\partial_\alpha v^r \partial^\alpha v_r - G_{rs} H^r_{\mu\alpha\beta} H^s{}_\nu{}^{\alpha\beta} = 0 \quad . \tag{2.11}$$

To this order, this amounts to a proof of supersymmetry, and it is also possible to show that the commutator of two supersymmetry transformations on the bosonic fields closes on the local symmetries:

$$\begin{aligned}
&[\delta_1, \delta_2] = \delta_{gct}(\xi^\mu = -i(\bar{\epsilon}_1 \gamma^\mu \epsilon_2)) + \delta_{tens}(\Lambda^r_\mu = -\frac{1}{2} v^r \xi_\mu - \xi^\nu B^r_{\mu\nu}) \\
&+ \delta_{SO(n)}(A^{mn} = \xi^\mu x^{mr}(\partial_\mu x_r^n)) + \delta_{Lorentz}(\Omega^{ab} = -\xi_\mu(\omega^{\mu ab} - v_r H^{r\mu ab})) \quad .
\end{aligned} \tag{2.12}$$

To this order, one can not see the local supersymmetry transformation in the gauge algebra, since the expected parameter, $\xi^\mu \Psi_\mu$, is generated by bosonic variations. As usual, the spin connection satisfies its equation of motion, that to lowest order in the fermi fields is

$$D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = 0 \quad , \tag{2.13}$$

and implies the absence of torsion.

Completing these equations will require terms cubic in the fermi fields in the fermionic equations, and terms quadratic in the fermi fields in their supersymmetry transformations. Supersymmetry will then determine corresponding modifications of the bosonic equations, and the (anti)self-duality conditions (2.5) will also be modified by terms quadratic in the fermi fields. Supercovariance actually fixes all terms containing the gravitino in the first-order equations and in the supersymmetry variations of fermi fields.

The supercovariant forms

$$\hat{\omega}_{\mu\nu\rho} = \omega^0_{\mu\nu\rho} - \frac{i}{2}(\bar{\Psi}_\mu \gamma_\nu \Psi_\rho + \bar{\Psi}_\nu \gamma_\rho \Psi_\mu + \bar{\Psi}_\nu \gamma_\mu \Psi_\rho) \quad , \tag{2.14}$$

$$\begin{aligned} \hat{H}^r_{\mu\nu\rho} &= H^r_{\mu\nu\rho} - \frac{1}{2} x^{mr} (\bar{\chi}^m \gamma_{\mu\nu} \Psi_\rho + \bar{\chi}^m \gamma_{\nu\rho} \Psi_\mu + \bar{\chi}^m \gamma_{\rho\mu} \Psi_\nu) \\ &- \frac{i}{2} v^r (\bar{\Psi}_\mu \gamma_\nu \Psi_\rho + \bar{\Psi}_\nu \gamma_\rho \Psi_\mu + \bar{\Psi}_\rho \gamma_\mu \Psi_\nu) \quad , \end{aligned} \tag{2.15}$$

$$\hat{\partial_\mu v^r} = \partial_\mu v^r - x^{mr} (\bar{\chi}^m \Psi_\mu) \quad , \tag{2.16}$$

where

$$\omega^0_{\mu\nu\rho} = \frac{1}{2} e_{\rho a} (\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a) - \frac{1}{2} e_{\mu a} (\partial_\nu e_\rho{}^a - \partial_\rho e_\nu{}^a) + \frac{1}{2} e_{\nu a} (\partial_\rho e_\mu{}^a - \partial_\mu e_\rho{}^a) \tag{2.17}$$

is the standard spin connection in the absence of torsion, do not generate derivatives of the parameter under supersymmetry. In the same spirit, one can consider the supercovariant transformations

$$\begin{aligned} \delta \Psi_\mu &= \hat{D}_\mu \epsilon + \frac{1}{4} v_r \hat{H}^r_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon \quad , \\ \delta \chi^m &= \frac{i}{2} x^m_r (\hat{\partial_\mu v^r}) \gamma^\mu \epsilon + \frac{i}{12} x^m_r \hat{H}^r_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon \quad . \end{aligned} \tag{2.18}$$

The tensorino transformation is complete, while the gravitino transformation could include additional terms quadratic in the tensorini. On the other hand, one does not expect modifications of the bosonic transformations in the complete theory.

2.1 Complete Supersymmetry Algebra

The algebra (2.12) has been obtained varying only the fermi fields in the bosonic supersymmetry transformations. The next step is to compute the commutator varying the bosonic fields as well. There is no important novelty in the complete commutator on v^r and on the vielbein $e_\mu{}^a$. However, the local Lorentz parameter is modified and takes the form

$$\Omega^{ab} = -\xi^\mu (\hat{\omega}_\mu{}^{ab} - v_r \hat{H}^{r\,ab}_\mu) \tag{2.19}$$

while, as anticipated, the supersymmetry parameter is

$$\zeta = \xi^\mu \Psi_\mu \quad . \tag{2.20}$$

These results are obtained using the torsion equation for $\hat{\omega}$,

$$\hat{D}_\mu e_\nu{}^a - \hat{D}_\nu e_\mu{}^a = 2S^a{}_{\mu\nu} = -i(\bar{\Psi}_\mu \gamma^a \Psi_\nu) \quad . \tag{2.21}$$

One can also compute the commutator on x^m_r. Eqs. (2.3) determine its supersymmetry variation

$$\delta x^m_r = v_r(\bar{\epsilon}\chi^m) \quad , \tag{2.22}$$

and the resulting commutator includes a local $SO(n)$ transformation of parameter

$$A^{mn} = \xi^\mu x^{mr}(\partial_\mu x^n_r) + (\bar{\chi}^m \epsilon_2)(\bar{\chi}^n \epsilon_1) - (\bar{\chi}^m \epsilon_1)(\bar{\chi}^n \epsilon_2) \quad . \tag{2.23}$$

New results come from the complete commutator on $B^r_{\mu\nu}$, where one needs to use the (anti)self-duality conditions. Supercovariantization is at work here, since these conditions are first-order equations, that become

$$G_{rs}\hat{H}^s_{\mu\nu\rho} = \frac{1}{6e}\epsilon_{\mu\nu\rho\alpha\beta\gamma}\hat{H}^{\alpha\beta\gamma}_r \quad . \tag{2.24}$$

It is actually possible to alter these conditions demanding that the modified tensor

$$\hat{\mathcal{H}}^r_{\mu\nu\rho} = \hat{H}^r_{\mu\nu\rho} + i\alpha v^r(\bar{\chi}^m \gamma_{\mu\nu\rho}\chi^m) \tag{2.25}$$

satisfy (anti)self-duality conditions as in eq. (2.24). Using eqs. (2.3), one can see that the new χ^2 terms contribute only to the self-duality condition, while the tensors $x^m_r \hat{H}^r_{\mu\nu\rho}$ remain antiself dual without extra χ^2 terms. Consequently, since the commutator on $B^r_{\mu\nu}$ uses only the antiself-duality conditions, the result does not contain terms proportional to α. The commutator on the tensor fields generates all local symmetries in the proper form, aside from the extra terms

$$[\delta_1, \delta_2]_{extra} B^r_{\mu\nu} = \frac{1}{2} v^r (\bar{\epsilon}_1 \chi^m)(\bar{\chi}^m \gamma_{\mu\nu} \epsilon_2) - \frac{1}{2} v^r (\bar{\epsilon}_2 \chi^m)(\bar{\chi}^m \gamma_{\mu\nu} \epsilon_1) \quad , \tag{2.26}$$

that may be canceled adding χ^2 terms to the transformation of the gravitino. The most general expression one can add is

$$\delta' \Psi_\mu = ia \; \gamma_\mu \chi^m (\bar{\epsilon} \chi^m) + ib \; \gamma_\nu \chi^m (\bar{\epsilon} \gamma_\mu{}^\nu \chi^m) + ic \; \gamma_{\mu\nu\rho} \chi^m (\bar{\epsilon} \gamma^{\nu\rho} \chi^m) \quad , \tag{2.27}$$

with a, b and c real coefficients, and the total commutator on $B^r_{\mu\nu}$ then leads to the relations

$$a + b = -\frac{1}{2} \quad , \qquad b + 2c = 0 \quad . \tag{2.28}$$

The commutator on $e_\mu{}^a$ now closes with a local Lorentz parameter modified by the addition of

$$\Delta \Omega^{ab} = -\frac{1}{2} [(\bar{\chi}^m \epsilon_1)(\bar{\epsilon}_2 \gamma^{ab} \chi^m) - (\bar{\chi}^m \epsilon_2)(\bar{\epsilon}_1 \gamma^{ab} \chi^m)] \quad , \tag{2.29}$$

while the commutators on the scalar fields are not modified.

One can now start to compute the commutators on fermi fields, that as usual close only on shell. Following [16], we will actually use this result to derive the complete fermionic equations. Let us begin with the commutator on the tensorini, using eq. (2.18). This fixes the free parameter in the gravitino variation and the parameter α in eq. (2.25), so that

$$a = -\frac{3}{8} \quad , \qquad b = -\frac{1}{8} \quad , \qquad c = \frac{1}{16} \quad , \qquad \alpha = -\frac{1}{8} \quad . \tag{2.30}$$

Supercovariance determines the field equation of the tensorini up to a term proportional to χ^3. Closure of the algebra fixes this additional term, and the end result is

$$\begin{aligned} &\gamma^\mu \hat{D}_\mu \chi^m - \frac{1}{12} v_r \hat{H}^r_{\mu\nu\rho} \gamma^{\mu\nu\rho} \chi^m - \frac{i}{2} x^m_r \hat{H}^{r\mu\nu\rho} \gamma_{\mu\nu} \Psi_\rho \\ &- \frac{i}{2} x^m_r (\hat{\partial_\nu v^r}) \gamma^\mu \gamma^\nu \Psi_\mu - \frac{i}{2} \gamma^\alpha \chi^n (\bar{\chi}^n \gamma_\alpha \chi^m) = 0 \quad . \end{aligned} \tag{2.31}$$

The complete commutator of two supersymmetry transformations on the tensorini is then

$$[\delta_1, \delta_2] \chi^m = \delta_{gct} \; \chi^m + \delta_{Lorentz} \; \chi^m + \delta_{SO(n)} \; \chi^m + \delta_{susy} \; \chi^m + \frac{1}{4} \gamma^\alpha \xi_\alpha \; [\text{eq. } \chi^m] \quad . \tag{2.32}$$

A similar result can be obtained for the gravitino. In this case the complete equation,

$$\begin{aligned}
&\gamma^{\mu\nu\rho}\hat{D}_\nu\Psi_\rho + \frac{1}{4}v_r\hat{H}^r_{\nu\alpha\beta}\gamma^{\mu\nu\rho}\gamma^{\alpha\beta}\Psi_\rho - \frac{i}{2}x^m_r\hat{H}^{r\mu\nu\rho}\gamma_{\nu\rho}\chi^m + \frac{i}{2}x^m_r(\hat{\partial_\nu v^r})\gamma^\nu\gamma^\mu\chi^m \\
&+\frac{3i}{2}\gamma^{\mu\alpha}\chi^m(\bar{\chi}^m\Psi_\alpha) - \frac{i}{4}\gamma^{\mu\alpha}\chi^m(\bar{\chi}^m\gamma_{\alpha\beta}\Psi^\beta) + \frac{i}{4}\gamma_{\alpha\beta}\chi^m(\bar{\chi}^m\gamma^{\mu\alpha}\Psi^\beta) \\
&-\frac{i}{2}\chi^m(\bar{\chi}^m\gamma^{\mu\alpha}\Psi_\alpha) = 0 \quad ,
\end{aligned} \tag{2.33}$$

is fixed by supercovariance, and the commutator closes up to terms proportional to a particular combination of eq. (2.33) and its γ-trace. Moreover, a non-trivial symplectic structure makes its first appearance in a commutator, so that the final result is

$$\begin{aligned}
&[\delta_1,\delta_2]\Psi^a_\mu = \delta_{gct}\Psi^a_\mu + \delta_{Lorentz}\Psi^a_\mu + \delta_{susy}\Psi^a_\mu \\
&+\frac{3}{8}\xi^\alpha\gamma_\alpha([eq.\ \Psi_\mu] - \frac{1}{4}\gamma_\mu[\gamma - trace])^a \\
&+\frac{1}{96}\sigma^{ia}_b\gamma^{\alpha\beta\gamma}\xi^i_{\alpha\beta\gamma}([eq.\ \Psi_\mu] - \frac{1}{4}\gamma_\mu[\gamma - trace])^b \quad ,
\end{aligned} \tag{2.34}$$

where

$$\xi^i_{\alpha\beta\gamma} = -i[\bar{\epsilon}_1\gamma_{\alpha\beta\gamma}\epsilon_2]^i \quad . \tag{2.35}$$

Summarizing, from the algebra we have obtained the complete fermionic equations of $(1,0)$ six-dimensional supergravity coupled to n tensor multiplets. In addition, the modified 3-form

$$\hat{\mathcal{H}}^r_{\mu\nu\rho} = \hat{H}^r_{\mu\nu\rho} - \frac{i}{8}v^r(\bar{\chi}^m\gamma_{\mu\nu\rho}\chi^m) \tag{2.36}$$

satisfies the (anti)self-duality conditions

$$G_{rs}\hat{\mathcal{H}}^s_{\mu\nu\rho} = \frac{1}{6e}\epsilon_{\mu\nu\rho\alpha\beta\gamma}\hat{\mathcal{H}}^{\alpha\beta\gamma}_r \quad . \tag{2.37}$$

We have also identified the complete supersymmetry transformations, that we collect here for convenience:

$$\begin{aligned}
&\delta e_\mu{}^a = -i(\bar{\epsilon}\gamma^a\Psi_\mu) \quad , \\
&\delta B^r_{\mu\nu} = iv^r(\bar{\Psi}_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2}x^{mr}(\bar{\chi}^m\gamma_{\mu\nu}\epsilon) \quad , \\
&\delta v_r = x^m_r(\bar{\chi}^m\epsilon) \quad , \\
&\delta\Psi_\mu = \hat{D}_\mu\epsilon + \frac{1}{4}v_r\hat{H}^r_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon - \frac{3i}{8}\gamma_\mu\chi^n(\bar{\epsilon}\chi^n) - \frac{i}{8}\gamma^\nu\chi^n(\bar{\epsilon}\gamma_{\mu\nu}\chi^n) + \frac{i}{16}\gamma_{\mu\nu\rho}\chi^n(\bar{\epsilon}\gamma^{\nu\rho}\chi^n) \quad , \\
&\delta\chi^m = \frac{i}{2}x^m_r(\hat{\partial_\alpha v^r})\gamma^\alpha\epsilon + \frac{i}{12}x^m_r\hat{H}^r_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\epsilon \quad .
\end{aligned} \tag{2.38}$$

2.2 Bosonic Equations of Motion and Supersymmetry

In order to obtain the bosonic equations, it is convenient to associate the fermionic equations to the Lagrangian

$$
\begin{aligned}
e^{-1}\mathcal{L}_{fer} \quad &= -\frac{i}{2}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_\nu[\frac{1}{2}(\omega+\hat{\omega})]\Psi_\rho - \frac{i}{8}v_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho) \\
&+\frac{i}{48}v_r[H+\hat{H}]^r_{\alpha\beta\gamma}(\bar{\Psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma}\Psi_\nu) + \frac{i}{2}\bar{\chi}^m\gamma^\mu D_\mu(\hat{\omega})\chi^m \\
&-\frac{i}{24}v_r\hat{H}^r_{\mu\nu\rho}(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m) + \frac{1}{4}x^m_r[\partial_\nu v^r + \hat{\partial_\nu v^r}](\bar{\Psi}_\mu\gamma^\nu\gamma^\mu\chi^m) \\
&-\frac{1}{8}x^m_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\Psi}_\mu\gamma_{\nu\rho}\chi^m) + \frac{1}{24}x^m_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\Psi}^\alpha\gamma_{\alpha\mu\nu\rho}\chi^m) \\
&+\frac{1}{8}(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m)(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho) - \frac{1}{8}(\bar{\chi}^m\gamma^\alpha\chi^n)(\bar{\chi}^m\gamma_\alpha\chi^n) \quad , \qquad (2.39)
\end{aligned}
$$

where, in the 1.5 order formalism, the spin connection

$$
\begin{aligned}
\omega_{\mu\nu\rho} &= \omega^0_{\mu\nu\rho} - \frac{i}{2}(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho + \bar{\Psi}_\nu\gamma_\rho\Psi_\mu + \bar{\Psi}_\nu\gamma_\mu\Psi_\rho) \\
&-\frac{i}{4}(\bar{\Psi}^\alpha\gamma_{\mu\nu\rho\alpha\beta}\Psi^\beta) - \frac{i}{4}(\bar{\chi}^m\gamma_{\mu\nu\rho}\chi^m) \qquad (2.40)
\end{aligned}
$$

satisfies its equation of motion, and is thus kept fixed in all variations.

In order to derive the bosonic equations, one can add to (2.39)

$$
e^{-1}\mathcal{L}_{bose} = -\frac{1}{4}R + \frac{1}{12}G_{rs}H^{r\mu\nu\rho}H^s_{\mu\nu\rho} - \frac{1}{4}\partial_\mu v^r\partial^\mu v_r \quad . \qquad (2.41)
$$

One can then obtain from $\mathcal{L}_{fer} + \mathcal{L}_{bose}$ the equations for the vielbein and the scalars, with the prescription that the (anti)self-duality conditions be used only after varying. Actually, ignoring momentarily eq. (2.37) and varying $\mathcal{L}_{fer} + \mathcal{L}_{bose}$ with respect to the antisymmetric tensor $B^r_{\mu\nu}$ yields the second-order tensor equation, the divergence of eq. (2.37),

$$
\begin{aligned}
D_\mu(G_{rs}\hat{H}^{s\mu\nu\rho}) &= \frac{1}{2}D_\mu[x^m_r(\bar{\chi}^m\gamma^{\mu\nu\rho\alpha}\Psi_\alpha)] \\
-\frac{i}{4}D_\mu[v_r(\bar{\Psi}_\alpha\gamma^{\alpha\beta\mu\nu\rho}\Psi_\beta)] &+ \frac{i}{4}D_\mu[v_r(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m)] \quad . \qquad (2.42)
\end{aligned}
$$

In a similar fashion, the scalar equation is

$$
\begin{aligned}
&x^m_r[\frac{1}{2}D_\mu(\partial^\mu v^r) + \frac{1}{3}v_s H^{r\mu\nu\rho}H^s_{\mu\nu\rho} - \frac{i}{4}H^{r\mu\nu\rho}(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho) + \frac{i}{24}H^r_{\alpha\beta\gamma}(\bar{\Psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma}\Psi_\nu) \\
&-\frac{i}{24}H^r_{\mu\nu\rho}(\bar{\chi}^n\gamma^{\mu\nu\rho}\chi^n) - \frac{1}{2}D_\nu(x^n_r(\bar{\Psi}_\mu\gamma^\nu\gamma^\mu\chi^n))] \\
&+v_r[-\frac{1}{4}H^{r\mu\nu\rho}(\bar{\Psi}_\mu\gamma_{\nu\rho}\chi^m) + \frac{1}{12}H^{r\mu\nu\rho}(\bar{\Psi}^\alpha\gamma_{\alpha\mu\nu\rho}\chi^m)] = 0 \quad , \qquad (2.43)
\end{aligned}
$$

while the Einstein equation is

$$
\begin{aligned}
&\frac{1}{2}e_{\beta a}[R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R-G_{rs}H^{r\alpha\nu\rho}H^{s\beta}{}_{\nu\rho}+\frac{1}{6}g^{\alpha\beta}G_{rs}H^{r}_{\mu\nu\rho}H^{s\mu\nu\rho}\\
&+\partial^\alpha v^r\partial^\beta v_r-\frac{1}{2}g^{\alpha\beta}\partial_\mu v^r\partial^\mu v_r]-\frac{i}{2}e^\alpha{}_a(\bar\Psi_\mu\gamma^{\mu\nu\rho}\hat D_\nu\Psi_\rho)+\frac{i}{2}(\bar\Psi_a\gamma^{\alpha\nu\rho}\hat D_\nu\Psi_\rho)\\
&+\frac{i}{2}(\bar\Psi_\mu\gamma^{\mu\alpha\rho}\hat D_a\Psi_\rho)+\frac{i}{2}(\bar\Psi_\mu\gamma^{\mu\nu\alpha}\hat D_\nu\Psi_a)-\frac{i}{4}e^\alpha{}_a v_r\hat H^r_{\mu\nu\rho}(\bar\Psi^\mu\gamma^\nu\Psi^\rho)+\frac{i}{4}v_r\hat H^r_{\mu a\rho}(\bar\Psi^\mu\gamma^\alpha\Psi^\rho)\\
&+\frac{i}{2}v_r\hat H^{r\alpha\nu\rho}(\bar\Psi_a\gamma_\nu\Psi_\rho)+\frac{i}{2}v_r\hat H^r_{a\nu\rho}(\bar\Psi^\alpha\gamma^\nu\Psi^\rho)+\frac{i}{24}e^\alpha{}_a v_r\hat H^r_{\beta\gamma\delta}(\bar\Psi_\mu\gamma^{\mu\nu\beta\gamma\delta}\Psi_\nu)\\
&-\frac{i}{12}v_r\hat H^r_{\beta\gamma\delta}(\bar\Psi_a\gamma^{\alpha\nu\beta\gamma\delta}\Psi_\nu)-\frac{i}{8}v_r\hat H^r_{a\beta\gamma}(\bar\Psi_\mu\gamma^{\mu\nu\alpha\beta\gamma}\Psi_\nu)+\frac{i}{2}e^\alpha{}_a(\bar\chi^m\gamma^\mu\hat D_\mu\chi^m)\\
&-\frac{i}{2}(\bar\chi^m\gamma^\alpha\hat D_a\chi^m)-\frac{i}{24}e^\alpha{}_a v_r\hat H^r_{\mu\nu\rho}(\bar\chi^m\gamma^{\mu\nu\rho}\chi^m)+\frac{i}{8}v_r\hat H^r_{a\nu\rho}(\bar\chi^m\gamma^{\alpha\nu\rho}\chi^m)\\
&+\frac{1}{2}e^\alpha{}_a x^m_r(\hat{\partial_\nu v^r})(\bar\Psi_\mu\gamma^\nu\gamma^\mu\chi^m)-\frac{1}{2}x^m_r(\hat{\partial_a v^r})(\bar\Psi_\mu\gamma^\alpha\gamma^\mu\chi^m)-\frac{1}{2}x^m_r(\hat{\partial_\nu v^r})(\bar\Psi_a\gamma^\nu\gamma^\alpha\chi^m)\\
&-\frac{1}{4}e^\alpha{}_a x^m_r\hat H^r_{\mu\nu\rho}(\bar\Psi^\mu\gamma^{\nu\rho}\chi^m)+\frac{1}{2}x^m_r\hat H^r_{\mu a\rho}(\bar\Psi^\mu\gamma^{\alpha\rho}\chi^m)+\frac{1}{4}x^m_r\hat H^{r\alpha}{}_{\nu\rho}(\bar\Psi_a\gamma^{\nu\rho}\chi^m)\\
&+\frac{1}{4}x^m_r\hat H^r_{a\nu\rho}(\bar\Psi^\alpha\gamma^{\nu\rho}\chi^m)+\frac{1}{12}e^\alpha{}_a x^m_r\hat H^r_{\mu\nu\rho}(\bar\Psi_\sigma\gamma^{\sigma\mu\nu\rho}\chi^m)-\frac{1}{12}x^m_r\hat H^r_{\mu\nu\rho}(\bar\Psi_a\gamma^{\alpha\mu\nu\rho}\chi^m)\\
&-\frac{1}{4}x^m_r\hat H^r_{a\nu\rho}(\bar\Psi_\sigma\gamma^{\sigma\alpha\nu\rho}\chi^m)+(fermi)^4=0 \quad . \qquad (2.44)
\end{aligned}
$$

For the sake of brevity, a number of quartic fermionic couplings, fully determined by the lagrangian of eqs. (2.39) and (2.41), are not written explicitly. It then takes a direct, if somewhat tedious, calculation to prove local supersymmetry, showing that

$$
\delta F\frac{\delta\mathcal{L}}{\delta F}+\delta B\frac{\delta\mathcal{L}}{\delta B}=0 \quad , \qquad (2.45)
$$

where F and B denote collectively the fermi and bose fields aside from the antisymmetric tensors. We would like to stress that the equations for the fermi fields defined from the gauge algebra differ from the lagrangian equations by overall factors that may be simply identified.

3 Inclusion of Vector Multiplets

A $(1,0)$ Yang-Mills multiplet in six dimensions comprises gauge vectors A_μ and pairs of left-handed spinors λ^a satisfying a symplectic Majorana condition, all in the adjoint representation of the gauge group. In this Section we write the complete field equations for $N=1$ supergravity coupled to n tensor multiplets and to vector multiplets, extending the results of [6, 7]. This setting plays a crucial role in six-dimensional perturbative type-I vacua, that naturally include a number of tensor multiplets [11], and more generally in the context of string dualities relating these to non-perturbative vacua of other strings and to M theory [13]. In all these cases, the anomaly polynomial comprises in principle an irreducible part, that in perturbative type-I vacua is removed by tadpole conditions, and a residual reducible part of the form

$$I_8 = - \sum_{x,y} c^r_x \; c^s_y \; \eta_{rs} \; \mathrm{tr}_x F^2 \; \mathrm{tr}_y F^2 \quad , \tag{3.1}$$

with the c's a collection of constants and η the Minkowski metric for $SO(1,n)$. In general, this residual anomaly should also include gravitational and mixed contributions, but we leave them aside, since they would contribute higher-derivative couplings not part of the low-energy effective supergravity.

The antisymmetric tensors are not inert under vector gauge transformations, as demanded by the Chern-Simons couplings

$$H^r = dB^r - c^{rz}\omega_z \quad , \tag{3.2}$$

where the index z runs over the various factors of the gauge group. Gauge invariance of H^r indeed requires that $B^r_{\mu\nu}$ transform under vector gauge transformations according to

$$\delta B^r = c^{rz} tr_z(\Lambda dA) \quad . \tag{3.3}$$

To lowest order, the (anti)self-duality conditions (2.5) are not affected, while their divergence becomes

$$D_\mu(G_{rs}H^{s\mu\nu\rho}) = -\frac{1}{4e}\epsilon^{\nu\rho\alpha\beta\gamma\delta} c^z_r tr_z(F_{\alpha\beta}F_{\gamma\delta}) \quad . \tag{3.4}$$

In a similar fashion, the fermionic equations become

$$\begin{aligned} &\gamma^{\mu\nu\rho} D_\nu \Psi_\rho + v_r H^{r\mu\nu\rho}\gamma_\nu \Psi_\rho - \frac{i}{2} x^m_r H^{r\mu\nu\rho}\gamma_{\nu\rho}\chi^m \\ &+\frac{i}{2} x^m_r \partial_\nu v^r \gamma^\nu \gamma^\mu \chi^m - \frac{1}{\sqrt{2}} v_r c^{rz} tr_z(F_{\sigma\tau}\gamma^{\sigma\tau}\gamma^\mu \lambda) = 0 \end{aligned} \tag{3.5}$$

for the gravitino,

$$\begin{aligned} &\gamma^\mu D_\mu \chi^m - \frac{1}{12} v_r H^{r\mu\nu\rho}\gamma_{\mu\nu\rho}\chi^m - \frac{i}{2} x^m_r H^{r\mu\nu\rho}\gamma_{\mu\nu}\Psi_\rho \\ &-\frac{i}{2} x^m_r \partial_\nu v^r \gamma^\mu \gamma^\nu \Psi_\mu - \frac{i}{\sqrt{2}} x^m_r c^{rz} tr_z(F_{\mu\nu}\gamma^{\mu\nu}\lambda) = 0 \end{aligned} \tag{3.6}$$

for the tensorini and

$$v_r c^{rz}\gamma^\mu D_\mu\lambda + \frac{1}{2}(\partial_\mu v_r)c^{rz}\gamma^\mu\lambda + \frac{1}{2\sqrt{2}}v_r c^{rz}F_{\alpha\beta}\gamma^\mu\gamma^{\alpha\beta}\Psi_\mu$$
$$+\frac{i}{2\sqrt{2}}x_r^m c^{rz}F_{\mu\nu}\gamma^{\mu\nu}\chi^m - \frac{1}{12}c^{rz}H_{r\mu\nu\rho}\gamma^{\mu\nu\rho}\lambda = 0 \tag{3.7}$$

for the gaugini. The supersymmetry transformations of the vector multiplet are

$$\delta A_\mu = -\frac{i}{\sqrt{2}}(\bar{\epsilon}\gamma_\mu\lambda) \quad ,$$
$$\delta\lambda = -\frac{1}{2\sqrt{2}}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \quad , \tag{3.8}$$

while the tensor transformation becomes

$$\delta B^r_{\mu\nu} = iv^r(\bar{\Psi}_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2}x^{mr}(\bar{\chi}^m\gamma_{\mu\nu}\epsilon) - 2c^{rz}tr_z(A_{[\mu}\delta A_{\nu]}) \quad . \tag{3.9}$$

The other transformations are not modified, aside from the change induced by (3.2) in the definition of H^r. Varying the fermi fields in the fermionic equations then gives the bosonic equations

$$x_r^m D_\mu(\partial^\mu v^r) + \frac{2}{3}x_r^m v_s H^r_{\alpha\beta\gamma}H^{s\alpha\beta\gamma} - x_r^m c^{rz}tr_z(F_{\alpha\beta}F^{\alpha\beta}) = 0 \tag{3.10}$$

for the scalar,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \partial_\mu v^r\partial_\nu v_r - \frac{1}{2}g_{\mu\nu}\partial_\alpha v^r\partial^\alpha v_r - G_{rs}H^r_{\mu\alpha\beta}H^s{}_\nu{}^{\alpha\beta}$$
$$+4v_r c^{rz}tr_z(F_{\alpha\mu}F^\alpha{}_\nu - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) = 0 \tag{3.11}$$

for the metric, and

$$D_\mu(v_r c^{rz}F^{\mu\nu}) - c^{rz}G_{rs}H^{s\nu\rho\sigma}F_{\rho\sigma} = 0 \tag{3.12}$$

for the vectors. The commutator of two supersymmetry transformations now includes a gauge transformation of parameter

$$\Lambda = \xi^\alpha A_\alpha \quad . \tag{3.13}$$

The novelty here is the non-vanishing divergence of eq. (3.12)

$$D_\mu J^\mu = -\frac{1}{2e}\epsilon^{\mu\nu\alpha\beta\gamma\delta}c^{rz}c_r^{z'}F_{\mu\nu}tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) \quad , \tag{3.14}$$

that reflects the presence of the residual gauge anomaly [7]. In particular, eq. (3.14) gives the covariant anomaly. Leaving aside momentarily the (anti)self-duality conditions, one might expect to derive eq. (3.12) from

$$e^{-1}\mathcal{L} = -\frac{1}{2}v_r c^{rz} tr_z F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}G_{rs}H^{r\mu\nu\rho}H^s_{\mu\nu\rho} \quad , \tag{3.15}$$

but this is actually not the case. In fact, eq. (3.12) is not integrable, while the inclusion of a Wess-Zumino term

$$e^{-1}\mathcal{L} = -\frac{1}{2}v_r c^{rz} tr_z F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}G_{rs}H^{r\mu\nu\rho}H^s_{\mu\nu\rho} - \frac{1}{8e}\epsilon^{\mu\nu\alpha\beta\gamma\delta}c^z_r B^r_{\mu\nu} Tr_z(F_{\alpha\beta}F_{\gamma\delta}) \quad , \tag{3.16}$$

turns the vector equation into

$$\begin{aligned} &D_\mu(v_r c^{rz}F^{\mu\nu}) - G_{rs}H^{s\nu\rho\sigma}c^{rz}F_{\rho\sigma} - \frac{1}{8e}\epsilon^{\nu\rho\alpha\beta\gamma\delta}c^z_r A_\rho c^{rz'} tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) \\ &-\frac{1}{12e}\epsilon^{\nu\rho\alpha\beta\gamma\delta}c^z_r F_{\rho\alpha}c^{rz'}\omega^{z'}_{\beta\gamma\delta} = 0 \quad , \end{aligned} \tag{3.17}$$

and now the divergence of the gauge current is the consistent anomaly [7]

$$\mathcal{A}_\Lambda = -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta\gamma\delta}c^z_r c^{rz'} tr_z(\Lambda\partial_\mu A_\nu)tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) \quad . \tag{3.18}$$

As an aside, one can observe that, ignoring the (anti)self-duality conditions, eq. (3.16) yields the second-order tensor equations (3.4) when varied with respect to the antisymmetric fields.

The Wess-Zumino consistency condition [17]

$$\delta_\Lambda \mathcal{A}_\epsilon = \delta_\epsilon \mathcal{A}_\Lambda \tag{3.19}$$

now implies the presence of a supersymmetry anomaly of the form

$$\mathcal{A}_\epsilon = -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta\gamma\delta}c^z_r c^{rz'} tr_z(\delta_\epsilon A_\mu A_\nu)tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) - \frac{1}{6}\epsilon^{\mu\nu\alpha\beta\gamma\delta}c^z_r c^{rz'} tr_z(\delta_\epsilon A_\mu F_{\nu\alpha})\omega^{z'}_{\beta\gamma\delta} \ , \tag{3.20}$$

and indeed the supersymmetry variation of the lagrangian is exactly eq. (3.20). Moreover, the divergence of the gravitino field equation, proportional to eq. (3.20), reflects the presence of the induced supersymmetry anomaly. We shall now complete this construction to all orders in the fermi fields.

3.1 Complete Supersymmetry Algebra

Let us begin by noting that the supercovariant Yang-Mills field strength is

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{i}{\sqrt{2}}(\bar{\lambda}\gamma_\mu\Psi_\nu) - \frac{i}{\sqrt{2}}(\bar{\lambda}\gamma_\nu\Psi_\mu) \quad , \tag{3.21}$$

while the other supercovariant fields are not modified. The supersymmetry transformations

$$\begin{aligned}
&\delta e_\mu{}^a = -i(\bar{\epsilon}\gamma^a\Psi_\mu) \quad , \\
&\delta B^r_{\mu\nu} = iv^r(\bar{\Psi}_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2}x^{mr}(\bar{\chi}^m\gamma_{\mu\nu}\epsilon) - 2c^{rz}tr_z(A_{[\mu}\delta A_{\nu]}) \quad , \\
&\delta v_r = x^m_r(\bar{\chi}^m\epsilon) \quad , \\
&\delta A_\mu = -\frac{i}{\sqrt{2}}(\bar{\epsilon}\gamma_\mu\lambda) \quad , \\
&\delta\Psi_\mu = \hat{D}_\mu\epsilon + \frac{1}{4}v_r\hat{H}^r_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon - \frac{3i}{8}\gamma_\mu\chi^n(\bar{\epsilon}\chi^n) - \frac{i}{8}\gamma^\nu\chi^n(\bar{\epsilon}\gamma_{\mu\nu}\chi^n) + \frac{i}{16}\gamma_{\mu\nu\rho}\chi^n(\bar{\epsilon}\gamma^{\nu\rho}\chi^n) \quad , \\
&\delta\chi^m = \frac{i}{2}x^m_r(\hat{\partial_\alpha v^r})\gamma^\alpha\epsilon + \frac{i}{12}x^m_r\hat{H}^r_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\epsilon \quad , \\
&\delta\lambda = -\frac{1}{2\sqrt{2}}\hat{F}_{\mu\nu}\gamma^{\mu\nu}\epsilon \quad ,
\end{aligned} \tag{3.22}$$

could in principle include additional terms proportional to λ^2. To be precise, one could add to $\delta\Psi$ a term proportional to $v_rc^{rz}tr_z(\lambda^2\epsilon)$, and to $\delta\chi$ a term proportional to $x^m_rc^{rz}tr_z(\lambda^2\epsilon)$. Moreover, the (anti)self-duality conditions could be modified by a self-dual term of the form $c^{rz}tr_z(\bar{\lambda}\gamma_{\mu\nu\rho}\lambda)$.

Let us proceed to study the supersymmetry algebra completely. On the scalar, the vielbein and the gauge field, the algebra closes with no subtleties, while additional information comes from the algebra on the tensor fields. Using the (anti)self-duality conditions satisfied by the 3-forms in eq. (2.25), one can show that the algebra on B^r closes up to the extra terms

$$[\delta_1, \delta_2]_{extra}B^r_{\mu\nu} = c^{rz}tr_z[(\bar{\epsilon}_1\gamma_\mu\lambda)(\bar{\epsilon}_2\gamma_\nu\lambda) - (\bar{\epsilon}_1\gamma_\nu\lambda)(\bar{\epsilon}_2\gamma_\mu\lambda)] \quad . \tag{3.23}$$

These can be canceled modifying the transformations of the gravitino and of the tensorini according to

$$\begin{aligned}
&\delta'\Psi_\mu = iv_rc^{rz}\{a\ tr_z[\lambda(\bar{\epsilon}\gamma_\mu\lambda)] + b\ tr_z[\gamma_{\mu\nu}\lambda(\bar{\epsilon}\gamma^\nu\lambda)] + c\ tr_z[\gamma^{\nu\rho}\lambda(\bar{\epsilon}\gamma_{\mu\nu\rho}\lambda)]\} \quad , \\
&\delta'\chi^m = d\ x^m_rc^{rz}tr_z[\gamma_\alpha\lambda(\bar{\epsilon}\gamma^\alpha\lambda)] \quad ,
\end{aligned} \tag{3.24}$$

and requiring that the modified 3-form

$$\hat{\mathcal{H}}^r_{\mu\nu\rho} = \hat{H}^r_{\mu\nu\rho} - \frac{i}{8} v^r (\bar{\chi}^m \gamma_{\mu\nu\rho} \chi^m) + i\alpha\ c^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) \tag{3.25}$$

satisfy the (anti)self-duality conditions

$$G_{rs} \hat{\mathcal{H}}^s_{\mu\nu\rho} = \frac{1}{6e} \epsilon_{\mu\nu\rho\alpha\beta\gamma} \hat{\mathcal{H}}^{\alpha\beta\gamma}_r \quad . \tag{3.26}$$

It should be appreciated that this change in the definition of the field strengths only affects the antiself-duality conditions, since $(\bar{\lambda}\gamma_{\alpha\beta\gamma}\lambda)$ is self-dual.

Requiring closure of the algebra on B^r then implies the conditions

$$\alpha = \frac{1}{4} \quad , \qquad d = \frac{1}{2} \quad , \qquad a + b = -1 \quad , \qquad b + 2c = 0 \quad , \tag{3.27}$$

and only one of the parameters is still undetermined. These terms have no effect for the scalars and the vectors, while the commutator on $e_\mu{}^a$ shows that the local Lorentz parameter is modified by the addition of

$$\Delta' \Omega^{ab} = v_r c^{rz} tr_z [(\bar{\epsilon}_1 \gamma^a \lambda)(\bar{\epsilon}_2 \gamma^b \lambda) - (\bar{\epsilon}_2 \gamma^a \lambda)(\bar{\epsilon}_1 \gamma^b \lambda)] \quad . \tag{3.28}$$

Turning to the fermi fields, the commutator on the tensorini χ^m involves techniques already met in the case with tensor multiplets only, and fixes the last free parameter in eqs. (3.27), so that

$$a = -\frac{9}{8} \quad , \qquad b = \frac{1}{8} \quad , \qquad c = -\frac{1}{16} \quad . \tag{3.29}$$

It closes on the field equation

$$\begin{aligned}
&\gamma^\mu \hat{D}_\mu \chi^m - \frac{1}{12} v_r \hat{H}^r_{\mu\nu\rho} \gamma^{\mu\nu\rho} \chi^m - \frac{i}{12} x^m_r \hat{H}^{r\mu\nu\rho} \gamma_\alpha \gamma_{\mu\nu\rho} \Psi^\alpha - \frac{i}{2} x^m_r (\hat{\partial_\nu v^r}) \gamma^\mu \gamma^\nu \Psi_\mu \\
&- \frac{i}{\sqrt{2}} x^m_r c^{rz} tr_z (\hat{F}_{\mu\nu} \gamma^{\mu\nu} \lambda) - \frac{1}{2} x^m_r c^{rz} tr_z [\gamma^\mu \gamma^\alpha \lambda (\bar{\Psi}_\mu \gamma_\alpha \lambda)] - \frac{i}{2} \gamma^\alpha \chi^n (\bar{\chi}^n \gamma_\alpha \chi^m) \\
&+ \frac{3i}{8} v_r c^{rz} tr_z [(\bar{\chi}^m \gamma_{\mu\nu} \lambda) \gamma^{\mu\nu} \lambda] + \frac{i}{4} v_r c^{rz} tr_z [(\bar{\chi}^m \lambda) \lambda] = 0 \quad ,
\end{aligned} \tag{3.30}$$

where all terms containing the gravitino are exactly determined by supercovariance. Moreover, the field equation appears in the commutator as in the theory without gauge fields:

$$[\delta_1, \delta_2] \chi^m = \delta_{gct} \chi^m + \delta_{Lorentz} \chi^m + \delta_{SO(n)} \chi^m + \delta_{susy} \chi^m + \frac{1}{4} \gamma^\alpha \xi_\alpha [eq.\ \chi^m] \quad . \tag{3.31}$$

Using similar techniques, one can compute the commutator on the gaugini λ. Here, however, the transformation

$$\delta\lambda = -\frac{1}{2\sqrt{2}}\hat{F}_{\mu\nu}\gamma^{\mu\nu}\epsilon \tag{3.32}$$

can not produce the terms proportional to x_r^m already present at the lowest order, and the only way to generate them is to modify eq. (3.32) by terms of the form

$$\frac{x_r^m c^{rz}}{v_s c^{sz}}\ \chi^m\ \lambda\ \epsilon \quad . \tag{3.33}$$

Singular couplings of this type were previously introduced in [8]. We therefore add all possible extra terms, that modulo Fierz identities are

$$\delta'\lambda = \frac{x_r^m c^{rz}}{v_s c^{sz}}[a(\bar{\chi}^m\lambda)\epsilon + b(\bar{\chi}^m\gamma_{\alpha\beta}\lambda)\gamma^{\alpha\beta}\epsilon + c(\bar{\chi}^m\epsilon)\lambda + d(\bar{\chi}^m\gamma_{\alpha\beta}\epsilon)\gamma^{\alpha\beta}\lambda] \quad , \tag{3.34}$$

and determine their coefficients from the algebra. Eq. (3.34) should not affect the vector (and, a fortiori, the tensor) commutator, and thus the coefficients are to obey the three equations

$$a - 2c = 0 \quad , \qquad b = 0 \quad , \qquad c + 2d = 0 \quad . \tag{3.35}$$

The other conditions,

$$a + 2b = -\frac{1}{2} \quad , \qquad c + 2d + 4b = 0 \quad , \qquad 2d + \frac{1}{8}a + \frac{1}{4}b = \frac{3}{16} \quad , \tag{3.36}$$

are obtained from the algebra on the gaugini, for instance tracking the terms generated by eq. (3.34) and proportional to ∂v. Combining eqs. (3.35) and (3.36), one finally obtains

$$a = -\frac{1}{2} \quad , \qquad c = -\frac{1}{4} \quad , \qquad d = \frac{1}{8} \quad . \tag{3.37}$$

As was the case for the gravitino already without vector multiplets, here the algebra generates the field equation with a non trivial symplectic structure,

$$\frac{3}{8}\gamma^\alpha\xi_\alpha[eq.\lambda^a] + \frac{1}{96}\gamma^{\alpha\beta\gamma}\sigma_b^{ia}\xi^i_{\alpha\beta\gamma}[eq.\lambda^b] \quad , \tag{3.38}$$

where $\xi^i_{\alpha\beta\gamma}$ is defined in eq. (2.35).

Eq. (3.34) also affects the algebra on the tensorini, whose field equation now includes two additional terms, and becomes

$$\gamma^\mu\hat{D}_\mu\chi^m - \frac{1}{12}v_r\hat{H}^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\chi^m - \frac{i}{12}x_r^m\hat{H}^{r\mu\nu\rho}\gamma_\alpha\gamma_{\mu\nu\rho}\Psi^\alpha - \frac{i}{2}x_r^m(\widehat{\partial_\nu v^r})\gamma^\mu\gamma^\nu\Psi_\mu$$

$$
\begin{aligned}
&-\frac{i}{\sqrt{2}}x^m_r c^{rz} tr_z(\hat{F}_{\mu\nu}\gamma^{\mu\nu}\lambda) - \frac{1}{2}x^m_r c^{rz} tr_z[\gamma^\mu\gamma^\alpha\lambda(\bar{\Psi}_\mu\gamma_\alpha\lambda)] - \frac{i}{2}\gamma^\alpha\chi^n(\bar{\chi}^n\gamma_\alpha\chi^m) \\
&+\frac{3i}{8}v_r c^{rz} tr_z[(\bar{\chi}^m\gamma_{\mu\nu}\lambda)\gamma^{\mu\nu}\lambda] + \frac{i}{4}v_r c^{rz} tr_z[(\bar{\chi}^m\lambda)\lambda] \\
&+\frac{3i}{2}\frac{x^m_r c^{rz} x^n_s c^{sz}}{v_t c^{tz}} tr_z[(\bar{\chi}^n\lambda)\lambda] - \frac{i}{4}\frac{x^m_r c^{rz} x^n_s c^{sz}}{v_t c^{tz}} tr_z[(\bar{\chi}^n\gamma_{\alpha\beta}\lambda)\gamma^{\alpha\beta}\lambda] = 0 \quad . \qquad (3.39)
\end{aligned}
$$

In the commutator of two supersymmetry transformations on the gaugini, these terms complete the algebra and let it close on the field equation, that now includes $\chi^2\lambda$ terms corresponding to the $\lambda^2\chi$ terms in the equation for the tensorini. In addition, the λ^3 terms comprise two groups: those proportional to $v_r v_s$ and those proportional to η_{rs} (recall, from eqs. (2.3), that $x^m_r x^m_s = v_r v_s - \eta_{rs}$). The former generate local Lorentz transformations according to eq. (3.28) and the term

$$
i v_r v_s c^{rz} c^{sz'} tr_{z'}[(\bar{\lambda}\gamma_\alpha\lambda')\gamma^\alpha\lambda'] \qquad (3.40)
$$

in the field equation, while the latter are

$$
\begin{aligned}
[\delta_1,\delta_2]_{extra}\lambda \quad &= \frac{c^z_r c^{rz'}}{v_s c^{sz}} tr_{z'}[-\frac{1}{4}(\bar{\epsilon}_1\gamma_\alpha\lambda')(\bar{\epsilon}_2\gamma_\beta\lambda')\gamma^{\alpha\beta}\lambda + \frac{1}{4}(\bar{\lambda}\gamma_\alpha\lambda')(\bar{\epsilon}_1\gamma^\alpha\lambda')\epsilon_2 - (1\leftrightarrow 2) \\
&+\frac{1}{16}(\bar{\epsilon}_1\gamma^\alpha\epsilon_2)(\bar{\lambda}'\gamma_{\alpha\beta\gamma}\lambda')\gamma^{\beta\gamma}\lambda] \quad . \qquad (3.41)
\end{aligned}
$$

In general, one could allow for a modified field equation including the λ^3 term

$$
-i\alpha c^z_r c^{rz'} tr_{z'}[(\bar{\lambda}\gamma_\alpha\lambda')\gamma^\alpha\lambda'] \quad , \qquad (3.42)
$$

with α an arbitrary parameter. Although the choice $\alpha = 1$ could seem the preferred one on account of the rigid limit, since the supersymmetric Yang-Mills theory in six dimensions does not contain such a λ^3 term, the $(1,0)$ supergravity is actually consistent for an arbitrary value of α, with the corresponding residual terms

$$
\begin{aligned}
\delta_{extra(\alpha)}\lambda \quad &\equiv [\delta_1,\delta_2]_{extra(\alpha)}\lambda = \frac{c^z_r c^{rz'}}{v_s c^{sz}} tr_{z'}[-\frac{1}{4}(\bar{\epsilon}_1\gamma_\alpha\lambda')(\bar{\epsilon}_2\gamma_\beta\lambda')\gamma^{\alpha\beta}\lambda \\
&-\frac{\alpha}{2}(\bar{\lambda}\gamma_\alpha\lambda')(\bar{\epsilon}_1\gamma_\beta\lambda')\gamma^{\alpha\beta}\epsilon_2 + \frac{\alpha}{16}(\bar{\lambda}\gamma_{\alpha\beta\gamma}\lambda')(\bar{\epsilon}_1\gamma^\gamma\lambda')\gamma^{\alpha\beta}\epsilon_2 \\
&+\frac{\alpha}{16}(\bar{\lambda}\gamma_\gamma\lambda')(\bar{\epsilon}_1\gamma^{\alpha\beta\gamma}\lambda')\gamma_{\alpha\beta}\epsilon_2 + \frac{1-\alpha}{4}(\bar{\lambda}\gamma_\alpha\lambda')(\bar{\epsilon}_1\gamma^\alpha\lambda')\epsilon_2 - (1\leftrightarrow 2) \\
&+\frac{1-\alpha}{16}(\bar{\epsilon}_1\gamma^\alpha\epsilon_2)(\bar{\lambda}'\gamma_{\alpha\beta\gamma}\lambda')\gamma^{\beta\gamma}\lambda] \qquad (3.43)
\end{aligned}
$$

in the commutator of two supersymmetry transformations on the gaugini. It should be appreciated that no choice of α can eliminate all these terms, that play the role of a central charge felt only by the gaugini. The Jacobi identity for this charge is properly satisfied for any value of α, and thus we are effectively discovering a 2-cocycle in our problem. It has long been known that, in general, anomalies in current conservations are accompanied by related anomalies in current commutators [18], but it is amusing to see how this "classically anomalous" model displays all these intricacies.

The complete algebra

$$
\begin{aligned}
[\delta_1,\delta_2]\lambda^a \quad &= \delta_{gct}\lambda^a + \delta_{Lorentz}\lambda^a + \delta_{susy}\lambda^a + \delta_{gauge}\lambda^a + \delta_{extra(\alpha)}\lambda^a \\
&+\frac{3}{8}\gamma^\alpha \xi_\alpha [eq.\lambda^a]_{(\alpha)} + \frac{1}{96}\gamma^{\alpha\beta\gamma}\sigma_b^{ia}\xi^i_{\alpha\beta\gamma}[eq.\lambda^b]_{(\alpha)}
\end{aligned} \tag{3.44}
$$

determines the complete field equation of the gaugini

$$
\begin{aligned}
& v_r c^{rz}\gamma^\mu \hat{D}_\mu \lambda + \frac{1}{2}(\hat{\partial_\mu v_r})c^{rz}\gamma^\mu\lambda + \frac{1}{2\sqrt{2}}v_r c^{rz}\hat{F}_{\alpha\beta}\gamma^\mu\gamma^{\alpha\beta}\Psi_\mu + \frac{i}{2\sqrt{2}}x_r^m c^{rz}\hat{F}_{\alpha\beta}\gamma^{\alpha\beta}\chi^m \\
&+\frac{1}{12}x_r^m c^{rz} x_s^m \hat{H}^s_{\mu\nu\rho}\gamma^{\mu\nu\rho}\lambda + \frac{1}{2}x_r^m c^{rz}(\bar{\chi}^m\lambda)\gamma^\mu\Psi_\mu + \frac{1}{4}x_r^m c^{rz}(\bar{\chi}^m\Psi_\mu)\gamma^\mu\lambda \\
&-\frac{1}{8}x_r^m c^{rz}(\bar{\chi}^m\gamma_{\alpha\beta}\Psi_\mu)\gamma^{\mu\alpha\beta}\lambda - \frac{1}{4}x_r^m c^{rz}(\bar{\chi}^m\gamma_{\mu\alpha}\Psi^\mu)\gamma^\alpha\lambda + \frac{i}{8}v_r c^{rz}(\bar{\lambda}\chi^m)\chi^m \\
&+\frac{3i}{16}v_r c^{rz}(\bar{\lambda}\gamma_{\alpha\beta}\chi^m)\gamma^{\alpha\beta}\chi^m + \frac{3i}{4}\frac{x_r^m c^{rz} x_s^n c^{sz}}{v_t c^{tz}}(\bar{\lambda}\chi^m)\chi^n - \frac{i}{8}\frac{x_r^m c^{rz} x_s^n c^{sz}}{v_t c^{tz}}(\bar{\lambda}\gamma_{\alpha\beta}\chi^m)\gamma^{\alpha\beta}\chi^n \\
&+iv_r v_s c^{rz}c^{sz'}tr_{z'}[(\bar{\lambda}\gamma_\alpha\lambda')\gamma^\alpha\lambda'] - i\alpha c_r^z c^{rz'}tr_{z'}[(\bar{\lambda}\gamma_\alpha\lambda')\gamma^\alpha\lambda'] = 0
\end{aligned} \tag{3.45}
$$

where, again, all terms containing the gravitino are fixed by supercovariance, while the $\chi^2\lambda$ terms are precisely as demanded by the $\lambda^2\chi$ terms in the field equations of the tensorini. At last, one can study the algebra on the gravitino, thus obtaining the field equation

$$
\begin{aligned}
& \gamma^{\mu\nu\rho}\hat{D}_\nu\Psi_\rho + \frac{1}{4}v_r\hat{H}^r_{\nu\alpha\beta}\gamma^{\mu\nu\rho}\gamma^{\alpha\beta}\Psi_\rho - \frac{i}{12}x_r^m\hat{H}^{r\alpha\beta\gamma}\gamma_{\alpha\beta\gamma}\gamma^\mu\chi^m + \frac{i}{2}x_r^m(\hat{\partial_\nu v^r})\gamma^\nu\gamma^\mu\chi^m \\
&+\frac{3i}{2}\gamma^{\mu\alpha}\chi^m(\bar{\chi}^m\Psi_\alpha) - \frac{i}{4}\gamma^{\mu\alpha}\chi^m(\bar{\chi}^m\gamma_{\alpha\beta}\Psi^\beta) + \frac{i}{4}\gamma_{\alpha\beta}\chi^m(\bar{\chi}^m\gamma^{\mu\alpha}\Psi^\beta) - \frac{i}{2}\chi^m(\bar{\chi}^m\gamma^{\mu\alpha}\Psi_\alpha) \\
&+v_r c^{rz}tr_z[-\frac{1}{\sqrt{2}}\gamma^{\alpha\beta}\gamma^\mu\lambda\hat{F}_{\alpha\beta} + \frac{3i}{4}\gamma^{\mu\nu\rho}\lambda(\bar{\Psi}_\nu\gamma_\nu\lambda) - \frac{i}{2}\gamma^\mu\lambda(\bar{\Psi}_\nu\gamma^\nu\lambda) + \frac{i}{2}\gamma^\nu\lambda(\bar{\Psi}_\nu\gamma^\mu\lambda) \\
&+\frac{i}{4}\gamma_\rho\lambda(\bar{\Psi}_\nu\gamma^{\mu\nu\rho}\lambda)] + \frac{1}{2}x_r^m c^{rz}tr_z[\gamma_\alpha\lambda(\bar{\chi}^m\gamma^\alpha\gamma^\mu\lambda)] = 0 \quad ,
\end{aligned} \tag{3.46}
$$

that enters the supersymmetry algebra as in eq. (2.34). Once more, all terms containing the gravitino are fixed by supercovariance, while the other $\lambda^2\chi$ terms are precisely as demanded by the $\lambda^2\Psi$ terms in the tensorino equation and by the $\lambda\Psi\chi$ terms in the equations of the gaugini.

Summarizing, from the algebra we have obtained the complete fermionic equations of $(1,0)$ six-dimensional supergravity coupled to vector and tensor multiplets. In addition, the modified 3-form

$$\hat{\mathcal{H}}^r_{\mu\nu\rho} = \hat{H}^r_{\mu\nu\rho} - \frac{i}{8} v^r (\bar{\chi}^m \gamma_{\mu\nu\rho} \chi^m) + \frac{i}{4} c^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) \tag{3.47}$$

satisfies the (anti)self-duality conditions

$$G_{rs} \hat{\mathcal{H}}^s_{\mu\nu\rho} = \frac{1}{6e} \epsilon_{\mu\nu\rho\alpha\beta\gamma} \hat{\mathcal{H}}^{\alpha\beta\gamma}_r \quad . \tag{3.48}$$

We have also identified the complete supersymmetry transformations, that we collect here for convenience:

$$\begin{aligned}
&\delta e_\mu{}^a = -i(\bar{\epsilon}\gamma^a \Psi_\mu) \quad , \\
&\delta B^r_{\mu\nu} = i v^r (\bar{\Psi}_{[\mu} \gamma_{\nu]} \epsilon) + \frac{1}{2} x^{mr} (\bar{\chi}^m \gamma_{\mu\nu} \epsilon) - 2 c^{rz} tr_z (A_{[\mu} \delta A_{\nu]}) \quad , \\
&\delta v_r = x^m_r (\bar{\chi}^m \epsilon) \quad , \\
&\delta A_\mu = -\frac{i}{\sqrt{2}} (\bar{\epsilon} \gamma_\mu \lambda) \quad , \\
&\delta \Psi_\mu = \hat{D}_\mu \epsilon + \frac{1}{4} v_r \hat{H}^r_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon - \frac{3i}{8} \gamma_\mu \chi^n (\bar{\epsilon} \chi^n) - \frac{i}{8} \gamma^\nu \chi^n (\bar{\epsilon} \gamma_{\mu\nu} \chi^n) + \frac{i}{16} \gamma_{\mu\nu\rho} \chi^n (\bar{\epsilon} \gamma^{\nu\rho} \chi^n) \\
&\qquad - \frac{9i}{8} v_r c^{rz} tr_z [\lambda (\bar{\epsilon} \gamma_\mu \lambda)] + \frac{i}{8} v_r c^{rz} tr_z [\gamma_{\mu\nu} \lambda (\bar{\epsilon} \gamma^\nu \lambda)] - \frac{i}{16} v_r c^{rz} tr_z [\gamma^{\nu\rho} \lambda (\bar{\epsilon} \gamma_{\mu\nu\rho} \lambda)] \quad , \\
&\delta \chi^m = \frac{i}{2} x^m_r (\hat{\partial_\alpha} v^r) \gamma^\alpha \epsilon + \frac{i}{12} x^m_r \hat{H}^r_{\alpha\beta\gamma} \gamma^{\alpha\beta\gamma} \epsilon + \frac{1}{2} x^m_r c^{rz} tr_z [\gamma_\alpha \lambda (\bar{\epsilon} \gamma^\alpha \lambda)] \quad , \\
&\delta \lambda = -\frac{1}{2\sqrt{2}} \hat{F}_{\mu\nu} \gamma^{\mu\nu} \epsilon - \frac{1}{2} \frac{x^m_r c^{rz}}{v_s c^{sz}} (\bar{\chi}^m \lambda) \epsilon - \frac{1}{4} \frac{x^m_r c^{rz}}{v_s c^{sz}} (\bar{\chi}^m \epsilon) \lambda \\
&\qquad + \frac{1}{8} \frac{x^m_r c^{rz}}{v_s c^{sz}} (\bar{\chi}^m \gamma_{\alpha\beta} \epsilon) \gamma^{\alpha\beta} \lambda \quad .
\end{aligned} \tag{3.49}$$

3.2 Bosonic Equations of Motion

Proceeding as in Section 2, the bosonic equations can be derived from a lagrangian, with the prescription of using the tensor (anti)self-duality conditions only after varying. The lagrangian is obtained supplementing $\mathcal{L}_{fermi} + \mathcal{L}_{bose}$ of eqs. (2.39) and (2.41) with the terms

$$
\begin{aligned}
&-\frac{1}{2}v_r c^{rz} tr_z(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{8e}\epsilon^{\mu\nu\alpha\beta\gamma\delta}c^z_r B^r_{\mu\nu} Tr_z(F_{\alpha\beta}F_{\gamma\delta}) \\
&+\frac{i}{2\sqrt{2}}v_r c^{rz} tr_z[(F+\hat{F})_{\sigma\delta}(\bar{\Psi}_\mu\gamma^{\sigma\delta}\gamma^\mu\lambda)] + \frac{1}{\sqrt{2}}x^m_r c^{rz} tr_z[(\bar{\chi}^m\gamma^{\mu\nu}\lambda)\hat{F}_{\mu\nu}] \\
&+iv_r c^{rz} tr_z[(\bar{\lambda}\gamma^m\hat{D}_\mu\lambda) + \frac{i}{12}x^m_r x^m_s \hat{H}^r_{\mu\nu\rho}c^{sz} tr_z(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda) + \frac{1}{16}v_r c^{rz} tr_z(\bar{\lambda}\gamma_{\mu\nu\rho}\lambda)(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m) \\
&-\frac{i}{8}(\bar{\chi}^m\gamma_{\mu\nu}\Psi_\rho)x^m_r c^{rz} tr_z(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda) - \frac{i}{2}x^m_r c^{rz} tr_z[(\bar{\chi}^m\gamma^\mu\gamma^\alpha\lambda)(\bar{\Psi}_\mu\gamma_\alpha\lambda)] \\
&-\frac{3}{16}v_r c^{rz} tr_z[(\bar{\chi}^m\gamma_{\mu\nu}\lambda)(\bar{\chi}^m\gamma^{\mu\nu}\lambda)] - \frac{1}{8}v_r c^{rz} tr_z[(\bar{\chi}^m\lambda)(\bar{\chi}^m\lambda) \\
&-\frac{3}{4}\frac{x^m_r c^{rz} x^n_s c^{sz}}{v_t c^{tz}} tr_z[(\bar{\chi}^m\lambda)(\bar{\chi}^n\lambda)] + \frac{1}{8}\frac{x^m_r c^{rz} x^n_s c^{sz}}{v_t c^{tz}} tr_z[(\bar{\chi}^m\gamma_{\alpha\beta}\lambda)(\bar{\chi}^n\gamma^{\alpha\beta}\lambda)] \\
&+\frac{1}{4}(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho)(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda) - \frac{1}{2}v_r v_s c^{rz} c^{sz'} tr_{z,z'}[(\bar{\lambda}\gamma_\alpha\lambda')(\bar{\lambda}\gamma^\alpha\lambda')] \\
&+\frac{\alpha}{2}c^{rz}c^{z'}_r tr_{z,z'}[(\bar{\lambda}\gamma_\alpha\lambda')(\bar{\lambda}\gamma^\alpha\lambda')] \quad ,
\end{aligned}
\tag{3.50}
$$

and the 1.5 order formalism requires that the spin connection $\omega_{\mu\nu\rho}$ now include the additional term

$$
\omega^{(\lambda)}_{\mu\nu\rho} = -\frac{i}{2}v_r c^{rz} tr_z(\bar{\lambda}\gamma_{\mu\nu\rho}\lambda) \quad .
\tag{3.51}
$$

With the new definition of ω, eqs. (2.39), (2.41) and (3.50) then yield the fermi equations. Moreover, varying with respect to $B^r_{\mu\nu}$ yields the second-order tensor equations, the divergence of the (anti)self-duality conditions. The vector equation is covariant, aside from the anomalous couplings introduced by the Wess-Zumino term in eq. (3.16). The complete residual gauge anomaly is thus given in eq. (3.18). As we shall see, it solves the Wess-Zumino consistency conditions even in the presence of supersymmetry.

The complete vector field equation is

$$
\begin{aligned}
&c^{rz}D_\nu(v_r F^{\nu\mu}) - G_{rs}\hat{H}^{r\mu\nu\rho}c^{sz}F_{\nu\rho} - \frac{1}{12e}\epsilon^{\mu\nu\rho\alpha\beta\gamma}c^z_r c^{rz'}F_{\nu\rho}\omega^{z'}_{\alpha\beta\gamma} \\
&-\frac{1}{8e}\epsilon^{\mu\nu\rho\alpha\beta\gamma}c^z_r c^{rz'}A_\nu tr_{z'}(F_{\rho\alpha}F_{\beta\gamma}) - \frac{i}{4}v_r c^{rz}F_{\nu\rho}(\bar{\Psi}_\alpha\gamma^{\alpha\beta\mu\nu\rho}\Psi_\beta) \\
&+\frac{i}{4}v_r c^{rz}F_{\nu\rho}(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m) - \frac{x^m_r c^{rz}}{2}F_{\nu\rho}(\bar{\Psi}_\alpha\gamma^{\alpha\mu\nu\rho}\chi^m) - \frac{i}{2}x^m_r x^m_s c^{rz}c^{sz'}F_{\nu\rho}tr_{z'}(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda) \\
&+\frac{i}{\sqrt{2}}c^{rz}D_\nu[v_r(\bar{\Psi}_\alpha\gamma^{\mu\nu}\gamma^\alpha\lambda)] + \frac{1}{\sqrt{2}}c^{rz}D_\nu[x^m_r(\bar{\chi}^m\gamma^{\mu\nu}\lambda)] = 0 \quad ,
\end{aligned}
\tag{3.52}
$$

the complete scalar equation is obtained adding to eq. (2.43) the terms

$$
x^m_r\{\frac{1}{32}c^{rz}tr_z(\bar{\lambda}\gamma_{\nu\alpha\beta}\lambda)(\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\gamma^{\alpha\beta}\Psi_\rho) + \frac{i}{2\sqrt{2}}c^{rz}tr_z[(\bar{\Psi}_\mu\gamma^{\alpha\beta}\gamma^\mu\lambda)(F+\hat{F})_{\alpha\beta})]
$$

$$
\begin{aligned}
&+ic^{rz}tr_z(\bar\lambda\gamma^\mu \hat D_\mu\lambda) - \frac{3}{16}c^{rz}tr_z[(\bar\chi^n\gamma^{\alpha\beta}\lambda)(\bar\chi^n\gamma_{\alpha\beta}\lambda)] - \frac{1}{8}c^{rz}tr_z[(\bar\chi^n\lambda)(\bar\chi^n\lambda)]\\
&+\frac{1}{4}c^{rz}tr_z(\bar\lambda\gamma^{\mu\nu\rho}\lambda)(\bar\Psi_\mu\gamma_\nu\Psi_\rho) - v_s c^{rz}c^{sz'}tr_{z,z'}[(\bar\lambda\gamma^\alpha\lambda')(\bar\lambda\gamma_\alpha\lambda')]\\
&-\frac{1}{8}\frac{x^n_s c^{sz}x^p_t c^{tz}}{(v\cdot c^z)^2}c^{rz}tr_z[(\bar\chi^n\gamma^{\alpha\beta}\lambda)(\bar\chi^p\gamma_{\alpha\beta}\lambda)] + \frac{3}{4}\frac{x^n_s c^{sz}x^p_t c^{tz}}{(v\cdot c^z)^2}c^{rz}tr_z[(\bar\chi^n\lambda)(\bar\chi^p\lambda)]\}\\
&+v_r\{\frac{1}{\sqrt2}c^{rz}tr_z[(\bar\chi^m\gamma^{\alpha\beta}\lambda)\hat F_{\alpha\beta}] + \frac{i}{12}x^m_s\hat H^{r\mu\nu\rho}c^{sz}tr_z(\bar\lambda\gamma_{\mu\nu\rho}\lambda)\\
&+\frac{i}{12}x^m_s\hat H^{s\mu\nu\rho}c^{rz}tr_z(\bar\lambda\gamma_{\mu\nu\rho}\lambda) - \frac{i}{8}c^{rz}tr_z(\bar\lambda\gamma_{\mu\nu\rho}\lambda)(\bar\chi^m\gamma^{\mu\nu}\Psi^\rho)\\
&-\frac{i}{2}c^{rz}tr_z[(\bar\chi^m\gamma^\mu\gamma^\alpha\lambda)(\bar\Psi_\mu\gamma_\alpha\lambda)] + \frac{1}{4}\frac{x^n_s c^{rz}c^{sz}}{v_t c^{tz}}tr_z[(\bar\chi^m\gamma^{\alpha\beta}\lambda)(\bar\chi^n\gamma_{\alpha\beta}\lambda)]\\
&-\frac{3}{2}\frac{x^n_s c^{rz}c^{sz}}{v_t c^{tz}}tr_z[(\bar\chi^m\lambda)(\bar\chi^n\lambda)]\} \quad ,
\end{aligned} \tag{3.53}
$$

and the Einstein equation is obtained adding to eq. (2.44) the terms

$$
\begin{aligned}
&c^{rz}tr_z\{2e_{\beta a}v_r(F_{\gamma\alpha}F^\gamma{}_\beta - \frac{1}{2}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}) + \frac{i}{\sqrt2}e^\alpha{}_a v_r(\bar\Psi_\mu\gamma^{\beta\gamma}\gamma^\mu\lambda)F_{\beta\gamma}\\
&-\frac{2i}{\sqrt2}v_r(\bar\Psi_\mu\gamma^{\alpha\gamma}\gamma^\mu\lambda)F_{a\gamma} - \frac{i}{\sqrt2}v_r(\bar\Psi_a\gamma^{\beta\gamma}\gamma^\alpha\lambda)F_{\beta\gamma} + \frac{1}{\sqrt2}x^m_r e^\alpha{}_a(\bar\chi^m\gamma^{\beta\gamma}\lambda)F_{\beta\gamma}\\
&-\sqrt2 x^m_r(\bar\chi^m\gamma^{\alpha\gamma}\lambda)F_{a\gamma} + ie^\alpha{}_a v_r(\bar\lambda\gamma^\mu D_\mu\lambda) - iv_r(\bar\lambda\gamma^\alpha D_a\lambda)\\
&+\frac{i}{12}e^\alpha{}_a x^m_r x^m_s H^s_{\mu\nu\rho}(\bar\lambda\gamma^{\mu\nu\rho}\lambda) - \frac{i}{4}x^m_r x^m_s H^s_{a\nu\rho}(\bar\lambda\gamma^{\alpha\nu\rho}\lambda)\\
&-\frac{i}{4}e_{\beta a}D_\rho[v_r(\bar\lambda\gamma^{\alpha\beta\rho}\lambda)] + (fermi)^4\} \quad .
\end{aligned} \tag{3.54}
$$

We would like to stress that this result is expressed in terms of the previous definition of ω, not corrected by bilinears in the gaugini. Moreover, for the sake of brevity, a number of quartic fermionic couplings, fully determined by the lagrangian of eqs. (2.39), (2.41) and (3.50), are not written explicitly. Letting F and B denote all the fermi and bose fields aside from the antisymmetric tensors, the supersymmetry variation of the lagrangian, after using the (anti)self-duality conditions of eq. (3.26), is

$$
\delta B\frac{\delta\mathcal{L}}{\delta B} + \delta F\frac{\delta\mathcal{L}}{\delta F} = \mathcal{A}_\epsilon \quad , \tag{3.55}
$$

where $\mathcal{A}_\epsilon$ is the complete supersymmetry anomaly. Neglecting the last term in eq. (3.50), *i.e.* setting α to zero,

$$
\begin{aligned}
\mathcal{A}_\epsilon \quad &= c^z_r c^{rz'}tr_{z,z'}\{-\frac{1}{4}\epsilon^{\mu\nu\alpha\beta\gamma\delta}\delta_\epsilon A_\mu A_\nu F'_{\alpha\beta}F'_{\gamma\delta} - \frac{1}{6}\epsilon^{\mu\nu\alpha\beta\gamma\delta}\delta_\epsilon A_\mu F_{\nu\alpha}\omega'_{\beta\gamma\delta}\\
&+\frac{ie}{2}\delta_\epsilon A_\mu F_{\nu\rho}(\bar\lambda'\gamma^{\mu\nu\rho}\lambda') + \frac{ie}{2}\delta_\epsilon A_\mu(\bar\lambda\gamma^{\mu\nu\rho}\lambda')F'_{\nu\rho} + ie\delta_\epsilon A_\mu(\bar\lambda\gamma_\nu\lambda')F'^{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{e}{32}\delta_\epsilon e_\mu{}^a(\bar\lambda\gamma^{\mu\alpha\beta}\lambda)(\bar\lambda'\gamma_{a\alpha\beta}\lambda') - \frac{e}{2\sqrt{2}}\delta_\epsilon A_\alpha(\bar\lambda\gamma^\alpha\gamma^\beta\gamma^\gamma\lambda')(\bar\lambda'\gamma_\beta\Psi_\gamma)\\
&+\frac{e x_s^m c^{sz'}}{v_t c^{tz'}}[-\frac{3i}{2\sqrt{2}}\delta_\epsilon A_\alpha(\bar\lambda\gamma^\alpha\lambda')(\bar\lambda'\chi^m) - \frac{i}{4\sqrt{2}}\delta_\epsilon A_\alpha(\bar\lambda\gamma^{\alpha\beta\gamma}\lambda')(\bar\lambda'\gamma_{\beta\gamma}\chi^m)\\
&-\frac{i}{2\sqrt{2}}\delta_\epsilon A_\alpha(\bar\lambda\gamma_\beta\lambda')(\bar\lambda'\gamma^{\alpha\beta}\chi^m)]\} \quad ,
\end{aligned}
\tag{3.56}
$$

while including the last term in eq. (3.50) would give the additional contribution

$$
\Delta\mathcal{A}_\epsilon = \delta_\epsilon\mathcal{L}_{\lambda^4} \quad , \tag{3.57}
$$

where

$$
\mathcal{L}_{\lambda^4} = \frac{e\alpha}{2}\; c_r^z c^{rz'} tr_{z,z'}[(\bar\lambda\gamma^\alpha\lambda')(\bar\lambda\gamma_\alpha\lambda')] \quad . \tag{3.58}
$$

In verifying the supersymmetry anomaly, the equations for the fermi fields and for the vector field are presented here must be rescaled by suitable overall factors that may be simply identified.

We now turn to show that $\mathcal{A}_\epsilon$ satisfies the complete Wess-Zumino consistency conditions.

3.3 Wess-Zumino Consistency Conditions

In general, the Wess-Zumino consistency conditions follow from the requirement that the symmetry algebra be realized on the effective action. For locally supersymmetric theories this implies

$$
\begin{aligned}
&\delta_{\Lambda_1}\mathcal{A}_{\Lambda_2} - \delta_{\Lambda_2}\mathcal{A}_{\Lambda_1} = \mathcal{A}_{[\Lambda_1,\Lambda_2]} \quad ,\\
&\delta_\epsilon\mathcal{A}_\Lambda = \delta_\Lambda\mathcal{A}_\epsilon \quad ,\\
&\delta_{\epsilon_1}\mathcal{A}_{\epsilon_2} - \delta_{\epsilon_2}\mathcal{A}_{\epsilon_1} = \mathcal{A}_{\tilde\epsilon} + \mathcal{A}_{\tilde\Lambda} \quad ,
\end{aligned}
\tag{3.59}
$$

where only gauge and supersymmetry anomalies are considered, and where $\tilde\epsilon$ and $\tilde\Lambda$ are the parameters of supersymmetry and gauge transformations determined by the supersymmetry algebra.

In global supersymmetry the analysis is somewhat simpler, since the r.h.s. of the last of eqs. (3.59) does not contain the (global) supersymmetry anomaly. Let us therefore begin by reviewing the case of supersymmetric Yang-Mills theory in four dimensions [19]. From the 6-form anomaly polynomial

$$I_6 = trF^3 \quad , \tag{3.60}$$

in the language of forms, one obtains the four-dimensional gauge anomaly

$$\mathcal{A}_\Lambda^{(4)} = tr[\Lambda(dA)^2 + \frac{ig}{2}\, d\Lambda A^3] \quad , \tag{3.61}$$

and from eqs. (3.59) one can determine the form of the global supersymmetry anomaly. With the classical lagrangian

$$\mathcal{L}_{SYM} = \mathrm{tr}\left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 2i\bar{\lambda}\gamma^\mu D_\mu \lambda\right] \quad , \tag{3.62}$$

and λ a right-handed Weyl spinor, the supersymmetry transformations are

$$\begin{aligned} \delta A_\mu &= \frac{i}{\sqrt{2}}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon) \quad , \\ \delta\lambda &= \frac{1}{2\sqrt{2}}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \quad . \end{aligned} \tag{3.63}$$

The second of eqs. (3.59) (with $\mathcal{A}_{\tilde{\epsilon}}$ absent in this global case), then determines the supersymmetry anomaly up to terms cubic in λ,

$$\mathcal{A}_\epsilon^{(4)} = tr[\delta_\epsilon A A(dA) + \delta_\epsilon A(dA)A - \frac{3ig}{2}\delta_\epsilon A A^3] \quad , \tag{3.64}$$

and indeed

$$\delta_{\epsilon_2}\mathcal{A}_{\epsilon_1}^{(4)} - \delta_{\epsilon_1}\mathcal{A}_{\epsilon_2}^{(4)} = \mathcal{A}_{\tilde{\Lambda}}^{(4)} + 3tr[\delta_{\epsilon_1}A\delta_{\epsilon_2}AF - \delta_{\epsilon_2}A\delta_{\epsilon_1}AF] \quad . \tag{3.65}$$

In order to compensate the second term in eq. (3.65), one is to add to $\mathcal{A}_\epsilon^{(4)}$ the gauge-invariant term

$$\Delta\mathcal{A}_\epsilon^{(4)} = -\frac{i}{2}tr[\delta_\epsilon A\bar{\lambda}\gamma^{(3)}\lambda + \bar{\lambda}\delta_\epsilon A\gamma^{(3)}\lambda] \quad , \tag{3.66}$$

so that $\mathcal{A}_\epsilon^{(4)} + \Delta\mathcal{A}_\epsilon^{(4)}$ is the proper global supersymmetry anomaly. Although the supersymmetry algebra closes only on the field equation of λ, in four dimensions a simple dimensional counting shows that eqs. (3.59) can not generate a term proportional to $\gamma^\mu D_\mu\lambda$. Therefore, in this case the Wess-Zumino consistency conditions close accidentally even off-shell, as pointed out in [19].

The situation is quite different in six dimensions. In this case, in the spirit of the previous Section, let us restrict our attention to the 8-form residual anomaly polynomial

$$I_8 = - \, c^{rz} c_r^{z'} tr_z(F^2) tr_{z'}(F^2) \quad , \tag{3.67}$$

where the sums are left implicit, so that the gauge anomaly is

$$\mathcal{A}_\Lambda^{(6)} = -c^{rz} c_r^{z'} tr_z(\Lambda dA) tr_{z'}(F^2) \quad . \tag{3.68}$$

Then, from the second of eqs. (3.59),

$$\mathcal{A}_\epsilon^{(6)} = -c^{rz} c_r^{z'} [tr_z(\delta_\epsilon A A) tr_{z'}(F^2) + 2 tr_z(\delta_\epsilon A F)\omega_3^{z'}] \quad , \tag{3.69}$$

but there are residual terms, so that

$$(\delta_{\epsilon_1} \mathcal{A}_{\epsilon_2}^{(6)} - \delta_{\epsilon_2} \mathcal{A}_{\epsilon_1}^{(6)})_{extra} = -4 c^{rz} c_r^{z'} [tr_z(\delta_{\epsilon_2} A \delta_{\epsilon_1} A) tr_{z'}(F^2) + 2 tr_z(\delta_{\epsilon_2} A F) tr_{z'}(\delta_{\epsilon_1} A F)] \quad . \tag{3.70}$$

Consequently, eq. (3.69) is to be modified by terms cubic in the gaugini, and the complete result, written in component notation, is finally

$$\begin{aligned}
\mathcal{A}_\epsilon^{(6)} \quad & = -\frac{1}{4} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu A_\nu) tr_{z'}(F'_{\alpha\beta} F'_{\gamma\delta}) \\
& -\frac{1}{6} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu F_{\nu\alpha}) \omega^{z'}_{\beta\gamma\delta} \\
& + A c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu F_{\nu\rho}) tr_{z'}(\bar{\lambda}' \gamma^{\mu\nu\rho} \lambda') \\
& + B c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu \bar{\lambda}) \gamma^{\mu\nu\rho} tr_{z'}(\lambda' F'_{\nu\rho}) \\
& + C c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu \bar{\lambda}) \gamma_\nu tr_{z'}(\lambda' F'^{\mu\nu}) \quad ,
\end{aligned} \tag{3.71}$$

where the coefficients A, B and C satisfy the relations

$$\begin{aligned}
& A + B = i \quad , \\
& C = 4A - 2B \quad .
\end{aligned} \tag{3.72}$$

These leave one undetermined parameter, in agreement with the well-known fact that anomalies are defined up to the variation of local functionals. Indeed, adding to the supersymmetry anomaly the term

$$\delta_\epsilon[(\bar{\lambda}\gamma^\alpha \lambda')(\bar{\lambda}\gamma_\alpha \lambda')] \tag{3.73}$$

corresponds to adding terms like the last three in eq. (3.71) with coefficients satisfying the relations $A+B=0$ and $C=4A-2B$, that thus preserve eqs. (3.72). One can then show that the last of eqs. (3.59) generates terms containing one derivative and four gaugini, that cancel using the Dirac equation $\gamma^\mu D_\mu \lambda = 0$. Naturally, something similar also happens in six-dimensional supergravity, as we are about to verify.

Returning to the supersymmetry anomaly of eq. (3.56), one can observe that the coefficients of the third, fourth and fifth terms are consistent with eqs. (3.72). Moreover, demanding that the last of eqs. (3.59) be satisfied fixes the other gauge-invariant terms to give exactly the anomaly in eq. (3.56). Finally, the Wess-Zumino condition is satisfied only on-shell, and one obtains

$$\begin{aligned}&(\delta_{\epsilon_1}\mathcal{A}_{\epsilon_2}-\delta_{\epsilon_2}\mathcal{A}_{\epsilon_1})=\mathcal{A}_{\tilde{\epsilon}}+\mathcal{A}_{\tilde{\Lambda}}+c^z_r c^{rz'} tr_{z,z'}\{-\frac{e}{8}\xi_\sigma(\bar{\lambda}\gamma_\mu\lambda')(\bar{\lambda}\gamma^\mu\gamma^\sigma[eq.\lambda']_{(\alpha=0)})\\&+\frac{e}{16}\xi_{i\sigma\delta\tau}\{[\bar{\lambda}\gamma^\tau\lambda']_i(\bar{\lambda}\gamma^{\sigma\delta}[eq.\lambda']_{(\alpha=0)})+[\bar{\lambda}\gamma^\tau\lambda]_i(\bar{\lambda}'\gamma^{\sigma\delta}[eq.\lambda']_{(\alpha=0)})\}\}\quad ,\end{aligned}\tag{3.74}$$

where we have stressed that the corresponding field equation for the gaugini is determined by eq. (3.50) with $\alpha=0$. To reiterate, the anomaly obtained for $\alpha=0$ naturally closes on the corresponding field equation for λ. Still, the identity

$$\begin{aligned}&c^z_r c^{rz'} tr_{z,z'}\{-\frac{e}{8}\xi_\sigma(\bar{\lambda}\gamma_\mu\lambda')(\bar{\lambda}\gamma^\mu\gamma^\sigma[eq.\lambda']_{(\alpha)})+\frac{e}{16}\xi_{i\sigma\delta\tau}\{[\bar{\lambda}\gamma^\tau\lambda']_i(\bar{\lambda}\gamma^{\sigma\delta}[eq.\lambda']_{(\alpha)})\\&+[\bar{\lambda}\gamma^\tau\lambda]_i(\bar{\lambda}'\gamma^{\sigma\delta}[eq.\lambda']_{(\alpha)})\}\}=c^z_r c^{rz'} tr_{z,z'}\{-\frac{e}{8}\xi_\sigma(\bar{\lambda}\gamma_\mu\lambda')(\bar{\lambda}\gamma^\mu\gamma^\sigma[eq.\lambda']_{(\alpha=0)})\\&+\frac{e}{16}\xi_{i\sigma\delta\tau}\{[\bar{\lambda}\gamma^\tau\lambda']_i(\bar{\lambda}\gamma^{\sigma\delta}[eq.\lambda']_{(\alpha=0)})+[\bar{\lambda}\gamma^\tau\lambda]_i(\bar{\lambda}'\gamma^{\sigma\delta}[eq.\lambda']_{(\alpha=0)})\}\}\\&+\frac{ie\alpha}{8}\frac{c^z_r c^{rz'} c^{z'}_s c^{sz''}}{v_t c^{tz'}}\xi_i^{\mu\nu\rho} tr_{z,z',z''}[\bar{\lambda}\gamma_\mu\lambda]_i[\bar{\lambda}'\gamma_\nu\lambda']_j[\bar{\lambda}''\gamma_\rho\lambda'']_j\end{aligned}\tag{3.75}$$

implies that the last term should somehow be generated in the anomaly, if the Wess-Zumino condition is to close for any value of α. In the presence of $\mathcal{L}_{\lambda^4}$, however, the anomaly is modified by eq. (3.57), and applying the last eqs. (3.59) to this term gives

$$\begin{aligned}[\delta_{\epsilon_1},\delta_{\epsilon_2}]\mathcal{L}_{\lambda^4}\quad&=([\delta_{\epsilon_1},\delta_{\epsilon_2}]e)\frac{\alpha}{2}c^z_r c^{rz'} tr_{z,z'}[(\bar{\lambda}\gamma_\alpha\lambda')(\bar{\lambda}\gamma^\alpha\lambda')]\\&+2e\alpha c^z_r c^{rz'} tr_{z,z'}[(\bar{\lambda}\gamma_\alpha\lambda')(\bar{\lambda}\gamma^\alpha[\delta_{\epsilon_1},\delta_{\epsilon_2}]\lambda')]\quad .\end{aligned}\tag{3.76}$$

The commutator in eq. (3.76) is fully known: in particular, the coordinate transformation in the second term combines with the commutator on e to give a total divergence, while gauge and local Lorentz transformations give a vanishing result. Moreover, the field equation is obtained from eq. (3.50). The charge in eq. (3.43) thus plays a crucial role: it generates in eq. (3.76) precisely

$$\frac{ie\alpha}{8}\frac{c^z_r c^{rz'} c^{z'}_s c^{sz''}}{v_t c^{tz'}}\xi^{\mu\nu\rho}_i tr_{z,z',z''}[\bar{\lambda}\gamma_\mu\lambda]_i[\bar{\lambda}'\gamma_\nu\lambda']_j[\bar{\lambda}''\gamma_\rho\lambda'']_j \quad , \tag{3.77}$$

as needed for consistency. Thus, one can understand the rationale behind the occurrence of the extension in the algebra on the gaugini: it lets the Wess-Zumino conditions close precisely on the field equations determined by the algebra. Since the Wess-Zumino conditions close on the equation of the gaugini, only these fields perceive the additional transformation.

4 Discussion

In the previous Sections we have completed the coupling of $(1,0)$ six-dimensional supergravity to tensor and vector multiplets. The coupling to tensor multiplets only, initiated by Romans [4], is of a more conventional nature, and parallels similar constructions in other supergravity models. One would expect similar results for the completion of the $(2,0)$ models in [4]. Our work is here confined to the field equations, but a lagrangian formulation of the (anti)self-dual two-forms is now possible, following the proposal of Pasti, Sorokin and Tonin [20] and indeed, while this work was being typed, results to this effect have been presented in a superspace formulation in [15]. On the other hand, the coupling to vector multiplets [6], originally suggested by perturbative type-I vacua [11], is of a more unconventional nature, since it is induced by the residual anomaly polynomial left over after tadpole conditions are imposed,

$$I_8 = \; - \sum_{x,y} \; c^r_x \; c^s_y \; \eta_{rs} \; \mathrm{tr}_x F^2 \; \mathrm{tr}_y F^2 \quad . \tag{4.1}$$

The corresponding Chern-Simons couplings of the two-forms,

$$H^r = dB^r - c^{rz}\omega_z \quad , \tag{4.2}$$

involve the constants c^r_z and determine related couplings of the other fields. In particular, the Yang-Mills currents are not conserved, and the consistent residual gauge anomaly is accompanied by a corresponding anomaly in the supersymmetry current [7]. In completing these results to all orders in the fermi fields, we have come to terms with another peculiar feature of anomalies, neatly displayed by these "classical" field equations: anomalous divergences of gauge currents are typically accompanied by corresponding anomalies in current commutators [18]. Indeed, we have discovered an amusing extension of the supersymmetry algebra on the gaugini, and we have linked its presence to an ambiguity in the definition of the supergravity model via Wess-Zumino consistency conditions. Whereas typical supergravity constructions yield a unique result, here one is free to add to the theory a quartic coupling for the gaugini

$$\mathcal{L}_{\lambda^4} = \frac{e\alpha}{2}\; c^z_r c^{rz'} tr_{z,z'}[(\bar{\lambda}\gamma^\alpha\lambda')(\bar{\lambda}\gamma_\alpha\lambda')] \quad , \tag{4.3}$$

whose presence affects only the supersymmetry anomaly. The Wess-Zumino conditions for six-dimensional supergravity close only on the field equation of the gaugini, and are consistent with any choice of α only thanks to the presence of the extension, as discussed in Section 3.3. Finally, we should mention that the singular gauge couplings $v^r c^z_r$ of ref. [6] are accompanied by corresponding divergent fermionic couplings, partly anticipated by Nishino and Sezgin [8].

Acknowledgment

We are grateful to Ya.S. Stanev for several stimulating discussions. The work of S.F. was supported in part by I.N.F.N., by DOE grant DE-FG03-91ER40662, Task C. and by E.E.C. Science Program SC1*CT92-0789.

5 Appendix: Notations and Conventions

In six dimensions, a 3-form $X_{\mu\nu\rho}$ is (anti)self-dual if

$$X_{\mu\nu\rho} = \;(-)\;\frac{1}{6e}\epsilon_{\mu\nu\rho\alpha\beta\gamma}X^{\alpha\beta\gamma} \quad . \tag{5.1}$$

If $X_{\mu\nu\rho}$ and $Y_{\mu\nu\rho}$ are both (anti)self-dual,

$$X_{\mu\nu\rho}Y^{\mu\nu\rho} = 0 \tag{5.2}$$

and

$$X_{\mu\nu\alpha}Y^{\mu\nu}{}_{\beta} - X_{\mu\nu\beta}Y^{\mu\nu}{}_{\alpha} = 0 \quad , \tag{5.3}$$

while if they have opposite duality properties

$$X_{\mu\nu\alpha}Y^{\mu\nu}{}_{\beta} + X_{\mu\nu\beta}Y^{\mu\nu}{}_{\alpha} = \frac{1}{3}g_{\alpha\beta}X_{\mu\nu\rho}Y^{\mu\nu\rho} \quad . \tag{5.4}$$

Moreover, an (anti)self-dual antisymmetric tensor $X_{\mu\nu\rho}$ satisfies

$$X^{\mu\nu\rho}X_{\alpha\beta\rho} = \frac{1}{4}[-\delta^{\mu}_{\beta}X_{\alpha\gamma\delta}X^{\nu\gamma\delta} + \delta^{\nu}_{\beta}X_{\alpha\gamma\delta}X^{\mu\gamma\delta} + \delta^{\mu}_{\alpha}X_{\beta\gamma\delta}X^{\nu\gamma\delta} - \delta^{\nu}_{\alpha}X_{\beta\gamma\delta}X^{\mu\gamma\delta}] \quad . \tag{5.5}$$

The signature of our metric is $(+,-,...,-)$, and the covariant derivative D_μ contains the full spin connection ω, the torsionless Christoffel connection, the gauge connection and the composite $SO(n)$ connection

$$S^{mn}_{\mu} = (\partial_\mu x^m_r)x^{nr} \quad . \tag{5.6}$$

The Clifford algebra is generated by $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, and the chirality matrix is

$$\gamma_7 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5 \quad . \tag{5.7}$$

Using $\epsilon^{012345} = +1$, one obtains

$$\gamma^{\mu_1\ldots\mu_n} = -\frac{(-1)^{[n/2]}}{(6-n)!e}\epsilon^{\mu_1\ldots\mu_n\nu_1\ldots\nu_{6-n}}\gamma_{\nu_1\ldots\nu_{6-n}}\gamma_7 \quad . \tag{5.8}$$

In particular, eq. (5.8) shows that $\gamma_{\mu\nu\rho}\Psi$ is self-dual if Ψ is left-handed, and antiself-dual if Ψ is right-handed.

Spinors are $Sp(2)$ doublets satisfying the symplectic Majorana condition

$$\Psi^a = \epsilon^{ab}C\bar{\Psi}^T_b \quad , \tag{5.9}$$

where

$$\bar{\Psi}_a = (\Psi^a)^\dagger\gamma_0 \tag{5.10}$$

and $\epsilon^{12} = \epsilon_{12} = 1$. The charge conjugation matrix is defined by

$$C\gamma^\mu C = -\gamma^{\mu,T} \quad , \tag{5.11}$$

where γ^0 is hermitian and the γ^i are anti-hermitian. Any bilinear $\bar{\Psi}_a\chi^b$ carries a pair of $Sp(2)$ indices, and can be decomposed in terms of the identity and of the three Pauli matrices. Indeed, one can form the bilinears

$$(\bar{\Psi}\chi) = \bar{\Psi}_a\chi^a \quad , \qquad [\bar{\Psi}\chi]_i = \sigma_{ia}{}^b\bar{\Psi}_b\chi^a \quad , \tag{5.12}$$

and standard properties imply that

$$\bar{\Psi}_a\chi^b = \frac{1}{2}\delta_a^b(\bar{\Psi}\chi) + \frac{1}{2}\sigma_{ia}{}^b[\bar{\Psi}\chi]_i \quad . \tag{5.13}$$

Using eq. (5.9), one can then see that the fermi bilinear $(\bar{\Psi}\chi)$ has standard behavior under Majorana-flip, namely

$$(\bar{\Psi}\chi) = (\bar{\chi}\Psi) \quad , \tag{5.14}$$

while all three bilinears $[\bar{\Psi}\chi]_i$ have the anomalous behavior

$$[\bar{\Psi}\chi]_i = -[\bar{\chi}\Psi]_i \quad . \tag{5.15}$$

Corresponding relations hold for all fermi bilinears, that naturally display pairs of opposite behaviors under Majorana flip. In particular, these properties imply that

$$[\bar{\Psi}\gamma_{\mu\nu\rho}\Psi]_i = 0 \quad , \tag{5.16}$$

a relation often used in deriving our results.

One can study Fierz relations between spinor bilinears using eq. (5.13). If Ψ and χ have the same chirality

$$\Psi^a\bar{\chi}_b = -\frac{1}{4}\bar{\chi}_b\gamma^\alpha\Psi^a\gamma_\alpha + \frac{1}{48}\bar{\chi}_b\gamma^{\alpha\beta\gamma}\Psi^a\gamma_{\alpha\beta\gamma} \quad , \tag{5.17}$$

while if they have opposite chirality

$$\Psi^a\bar{\chi}_b = -\frac{1}{4}\bar{\chi}_b\Psi^a + \frac{1}{8}\bar{\chi}_b\gamma^{\alpha\beta}\Psi^a\gamma_{\alpha\beta} \quad . \tag{5.18}$$

Interesting results obtain when one (anti)symmetrizes these relations. In particular, eq. (5.17) implies

$$\Psi^a\bar{\chi}_b - \chi^a\bar{\Psi}_b = -\frac{1}{4}(\bar{\chi}\gamma^\alpha\Psi)\delta_b^a\gamma_\alpha + \frac{1}{48}[\bar{\chi}\gamma^{\alpha\beta\gamma}\Psi]_i\sigma_{ib}{}^a\gamma_{\alpha\beta\gamma} \quad , \tag{5.19}$$

a result often used in the paper.

References

[1] L. Alvarez-Gaumé and E. Witten, *Nucl. Phys.* B234 (1983) 269.

[2] M.B. Green and J.H. Schwarz, *Phys. Lett.* **149B** (1984) 117.

[3] R. D'Auria, P. Fré and T. Regge, *Phys. Lett.* **128B** (1983) 44.

[4] L.J. Romans, *Nucl. Phys.* **B276** (1986) 71.

[5] H. Nishino and E. Sezgin, *Nucl. Phys.* **B278** (1986) 353.

[6] A. Sagnotti, *Phys. Lett.* **294B** (1992) 196.

[7] S. Ferrara, R. Minasian and A. Sagnotti, *Nucl. Phys.* **B474** (1996) 323.

[8] H. Nishino and E. Sezgin, hep-th/9703075.

[9] S. Randjbar-Daemi, A. Salam, E. Sezgin and J. Strathdee, *Phys. Lett.* **151B** (1985) 351.

[10] A. Sagnotti, *in* Cargese '87, "Non-Perturbative Quantum Field Theory", eds. G.Mack et al (Pergamon Press, 1988), p. 521.

[11] M. Bianchi and A. Sagnotti, *Phys. Lett.* **B247** (1990) 517; *Nucl. Phys.* **B361** (1991) 519;

E. Gimon and J. Polchinski, hep-th/9601038;

A. Dabholkar and J. Park, hep-th/9602030, 9604178;

E. Gimon and C.V. Johnson, hep-th/9604129;

M. Berkooz, R. Leigh, J. Polchinski, J.H. Schwarz, N. Seiberg and E. Witten, *Nucl. Phys.* **B475** (1996) 115;

J. Blum and A. Zaffaroni, *Phys. Lett.* **B387** (1996) 71;

J. Blum, *Nucl. Phys.* **B486** (1997) 34;

A. Sen, *Nucl. Phys.* **B475** (1996) 562, *Phys. Rev.* **D55** (1997) 7345, hep-th/9709159;

J. Blum and K. Intriligator, hep-th/9705030, 9705044;

J. Blum and K.R. Dienes, hep-th/9707148, 9707160;

K. Dasgupta and S. Mukhi, *Phys. Lett.* **B385** (1996) 125;

C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Ya.S. Stanev, *Phys. Lett.*

B387 (1996) 743;

For a review, see: L.E. Ibañez and A.M. Uranga, hep-th/9707075.

[12] N. Seiberg and E. Witten, *Nucl. Phys.* **B471** (1996) 121;
M.J. Duff, H. Lü and C. Pope, *Phys. Lett.* **B378** (1996) 101.

[13] M.J. Duff, R. Minasian and E. Witten, *Nucl. Phys.* **B475** (1996) 413.

[14] E. Witten, *Nucl. Phys.* **B443** (1995) 85.

[15] G. Dall'Agata, K. Lechner and M. Tonin, hep-th/9710127.

[16] J.H. Schwarz, *Nucl. Phys.* **B226** (1983) 269.

[17] J. Wess and B. Zumino, *Phys. Lett.* **B37** (1971) 95.

[18] See R. Jackiw's lectures in : S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, "Current Algebra and Anomalies" (World Scientific, 1985).

[19] G. Girardi, R. Grimm and R. Stora, *Phys. Lett.* **B156** (1985) 203;
L. Bonora, P. Pasti and M. Tonin, *Phys. Lett.* **B156** (1985) 341;
E. Guadagnini, K. Konishi and M. Mintchev, *Phys. Lett.* **B157** (1985) 37;
N.K. Nielsen, *Nucl. Phys.* **B244** (1984) 499;
H. Itoyama, V.P. Nair and H. Ren, *Nucl. Phys.* **B262** (1985) 317;
E. Guadagnini and M. Mintchev, *Nucl. Phys.* **B269** (1986) 543;
S. Ferrara, A. Masiero, M. Porrati and R. Stora, *Nucl. Phys.* **B417** (1994) 238.

[20] P. Pasti, D. Sorokin and M. Tonin, *Phys. Rev.* **D55** (1995) 6292.